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SOVIET INSTRUMENTATION AND
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Ya. Z. Tsypkin

LENIN PRIZE LAUREATE

On April 22, 1960, the Committee for the Lenin prize in the domain of science and technology for the Soviet Ministers of the Union of Soviet Socialist Republics awarded the Lenin prize to Doctor of Mechanics, Professor Yakov Zalmanovich Tsypkin for his work on the theory of sampled-data and relay automatic systems, presented in the monographs, Transient and Steady-State Processes in Sampled-Data Circuits, Theory of Relay Systems of Automatic Control, Theory of Sampled-Data Systems, published during the years 1951-1958.

The editorial board of the journal Avtomatika i Telemekhanika (Automation and Remote Control) warmly congratulates Yakov Zalmanovich Tsypkin for his high honor, and wishes him further creative successes.

The V. I. Lenin prize for works on the theory of relay and sampled-data systems was awarded by the director of the laboratory of the Institute for Automation and Remote Control of the Academy of Sciences of the USSR to Professor Yakov Zalmanovich Tsypkin, member of the editorial board of the journal Avtomatika i Telemekhanika (Automation and Remote Control) and doctor of engineering science.

For the first time, a cycle of theoretical works in the domain of automation has received the highest award. The Institute for Automation and Remote Control of the AN SSSR (Academy of Sciences of the USSR) is rightfully proud that the works singled out for the Lenin prize were executed by one of its leading scientific workers.

The editorial board of the journal Avtomatika i Telemekhanika (Automation and Remote Control) is particularly pleased to state that works of Ya. Z. Tsypkin, awarded this high honor, were published in this journal, starting in 1948.

The biography of Lenin prize laureate Ya. Z. Tsypkin is a typical biography of a young Soviet scientist and patriot of our fatherland.

Ya. Z. Tsypkin's first steps as a scientist were connected with the investigation of continuous control systems. The concept of degree of stability, suggested by him in conjunction with P. V. Bromberg [1, 4, 5, 8, 18, 48], entered into automatic control theory and was widely disseminated in the international literature. His following works were devoted to the dynamics of automatic control systems when lags were taken into account. Ya.Z. Tsypkin developed, and applied to the analysis of stability of these systems, frequency methods [4, 6, 7, 8]. The results which he obtained in this area underlay his doctoral dissertation, "Systems with lagging feedback," [9].

The methods developed by. Ya. Z. Tsypkin for investigating systems with lags received broad recognition, both in the Soviet Union and abroad. It is appropriate to recall here that, in evaluating the work of this period, Academician A. A. Andronov characterized Ya. Z. Tyspkin as one of the most vigorous and promising scientists working in the domain of control theory in the USSR.

The tremendous development of computing technology and radioelectronics in the postwar years engendered a heightened interest in discrete devices for control and communications, in particular, in relay, sampled-data and digital devices. Ya. Z. Tsypkin is one of the first scientists to have properly appreciated the importance of this trend and, more than ten years ago, he applied himself to the development of a general theory of discrete automatic systems. His first work in this direction dates back to 1948 [10].

The fundamentals of sampled-data system theory were first presented by him in [12, 13, 15]. The generality of the developed theories of sampled-data systems was not inferior to the generality of theories of continuous systems. In 1951, Ya. Z. Tsypkin published a monograph on sampled-data circuits, Transient and Steady-State Processes in Sampled-Data Circuit [17]. In [20] there was developed a frequency method of analyzing discontinuous control systems which allowed one to establish in which cases sampled-data control improves the system's dynamic properties as compared to continuous control. A further series of works were devoted to the effect of random stimuli [22], of pulse shapes [33], of forcing actions [40] and of nonlinearities [23, 38] on the dynamic properties of sampled-data systems.

Starting in 1956, he extended sampled-data system theory to cover systems with digital computing devices [39, 44, 46, 47, 49, 50, 51].

The enumerated works were widely used by various organizations, designers, developers, investigators and users of sampled-data systems. The developed theory of sampled-data systems entered into the VUZ (university) curriculum and into courses for advanced engineering [45]. It was widely used and further developed in a number of works by Soviet and foreign scientists.

In 1958, Ya. Z. Tsypkin published the fundamental monograph, Theory of Sampled-Data Systems, [52] which is devoted to the analysis and synthesis of sampled-data systems and contains the fundamental general theory of sampled-data automatic systems, applied in various domains of technology. Together with sampled-data systems, relay systems were investigated by him in a number of works. In the theory of relay systems, Ya. Z. Tsypkin developed general and precise methods of investigation, superficially similar to linear methods but, at the same time, applicable to this class of essentially nonlinear systems. These methods permitted the investigation of various modes of operation in relay systems, the comparison of types of circuits and the development of new, advanced schemes. These works [16, 19, 23, 27, 29, 31] were summarized in the monograph, Theory of Relay Systems of Automatic Control [34].

The monographs of Ya. Z. Tsypkin, and many of this papers, were published in the German Democratic Republic, China, Poland, Rumania, Czechoslovakia, England, France, the German Federal Republic, the United States of America, and Japan, and were given as papers at international congresses [43, 46, 50, 51].

It is necessary to mention the characteristic traits of the creative style of Ya. Z. Tsypkin – clarity and explicitness in posing the problem, simplicity and elegance of the method of investigation, tense and concentrated purposefulness.

It is no coincidence that his last monographs constitute three important successive steps in the solution of a single problem, that of creating a general theory of discrete automatic systems.

Ya. Z. Tsypkin devoted much attention and work to the question of preparing a professional cadre. He is widely known as a superb lecturer. His lectures on automatic control theory have been very popular [14, 21, 28, 45, 54].

Under Ya. Z. Tsypkin's editorship, the fundamental foreign works on automatic control have been translated

and published here (McColl, Fundamental Theory of Servomechanism (1947); Lauer, Lesnick, and Matson, Servomechanism Fundamentals (1948); James, Nichols, and Phillips, Theory of Servomechanisms (1949); Flugge-Lotz, Phase Plane Method in Relay System Theory, 1959; and others).

Under Ya. Z. Tsypkin's direction, a large number of dissertations were executed, comprising a broad circle of questions in radio engineering, electronics and automatic control.

Professor Ya. Z. Tsypkin is an important Soviet scientist, creator of a unified theory of sampled-data and relay automatic systems which has found wide practical application and has obtained world-wide recognition.

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2. "Stability of linear systems with feedback," Radiotekhnika No. 5 (1946).
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4. "Stability of automatic frequency control systems with account taken of delay effects," Radiotekhnika Nos. 7-8 (1946).
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ANALYTIC CONTROLLER DESIGN. II

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(Moscow)

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Original article submitted December 23, 1959

The solution is presented for the problem of analytic controller design in correspondence with a given optimizing functional. The solution is given for a closed region which contains the boundary of excursion of the controlling organ.

1. Optimal Systems, Defined in a Closed Region

Essential in all automatic control systems is the limitation on the excursion of the controlling organs ξ:

$$|\xi| \leqslant \bar{\xi} \tag{1.1}$$

on the limitation on the magnitude of the velocity of this excursion

$$|\dot{\xi}| \leqslant \bar{f}, \tag{1.2}$$

where $\bar{\xi}$, f are positive numbers.

With such limitations, the system becomes essentially nonlinear, since the presence of the equality sign in (1.1) and (1.2) permits the system to remain on the boundary of the region $\overline{N}$ of its definition. This region is closed. Moreover, it is impossible here to seek an optimal solution to the problem among functions of the class C_1, since there can be discontinuities on the boundary of the function which solves the problem. By taking into account what has just been said, we formulate the following variational problem.

We consider a closed system to be controlled, in which the disturbed motion of the controlled object is given by the equations

$$g_k = \dot{\eta}_k - \left(\sum_\alpha b_{k\alpha}\eta_\alpha + m_k\xi\right) = 0 \quad (k = 1, \ldots, n). \tag{1.3}$$

The notation in (1.3) has the same meaning as in [1]. It is assumed that (1.3), in conjunction with the controller equations being sought, are defined in the closed region $\overline{N}$, characterized by (1.1). In this region there are given the sole boundary conditions

$$\eta_{10} = \eta_1(0), \ldots, \eta_n(0) = \eta_{n0}, \ \xi(0) = \xi_0;$$
$$\eta_1(\infty) = \ldots = \eta_n(\infty) = \xi(\infty) = 0 \tag{1.4}$$

They mean simply this: that no matter what transient response arises in $\overline{N}$, it must terminate, for $t^* = \infty$, with the system at the origin of coordinates.

As the criterion of optimality, we take the integral

$$I = \int_0^\infty V dt, \tag{1.5}$$

where V is the positively defined quadratic form

$$V = \sum_k a_k \eta_k^2 + c\xi^2. \tag{1.6}$$

We shall search for such continuous functions ξ, η_1, ..., η_n in Class C (permitting, generally speaking, discontinuities of the first derivatives) which minimize the integral in (1.5).

The problem just formulated belongs to the class of the so-called discontinuous problems of the calculus of variations. To solve such problems there exist various methods of the classical variational calculus, as well as newer methods, presented in the works cited in [1]. Here we shall use the methods of the classical variational calculus, supplemented by a nonlinear transformation which, in optimal problems of the first class [1], was used in [2].*

2. Solution of the Problem

We consider the case when the limitation is as in (1.1), and we set

$$\xi = \varphi(\zeta). \tag{2.1}$$

We define the function $\varphi(\zeta)$ as

$$\varphi(\zeta) = \begin{cases} +\bar{\xi} & \text{for} \quad \zeta \geqslant \zeta^*, \\ \varphi(\zeta) & \text{for} \quad |\zeta| < \zeta^*, \\ -\bar{\xi} & \text{for} \quad \zeta \leqslant -\zeta^*, \end{cases} \tag{2.2}$$

where ζ^* is a given positive number and $\varphi(\zeta)$ is any continuous function whose derivative is continuous for $|\zeta| < \zeta^*$, and which is equal to zero for $\zeta = 0$. In particular, such a condition is satisfied by the function $\varphi(\zeta) = \bar{\xi} \sin \zeta$ for $\zeta^* = \pi/2$ (Fig. 1).

Transformations (2.1) and (2.2) translate the closed region $N(\xi, \eta_1, \ldots, \eta_n)$ into the open region $\overline{N}(\zeta, \eta_1, \ldots, \eta_n)$ and permit one to use the well-known methods of variational calculus for the solution of ordinary Lagrange problems. The only special feature of this solution is that one must verify that the Weierstrass-Erdmann conditions

* The author's attention was called to the transformation by I. A. Litovchenko, to whom the author expresses his gratitude.

hold at all points of discontinuity of the derivatives. To begin the solution of the problem, we set

$$H = V + \sum_k \lambda_k \left[\dot{\eta}_{ik} - \sum_\alpha b_{k\alpha}\eta_{i\alpha} - m_k\varphi(\zeta)\right]. \tag{2.3}$$

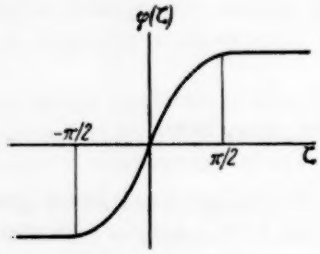

Fig. 1.

On the basis of (2.3), we find that

$$\frac{\partial H}{\partial \dot{\eta}_{ik}} = \lambda_k, \qquad \frac{\partial H}{\partial \eta_{ik}} = 2a_k\eta_{ik} - \sum_\alpha \lambda_\alpha b_{\alpha k},$$

$$\frac{\partial H}{\partial \dot{\zeta}} = 0, \qquad \frac{\partial H}{\partial \zeta} = \left[2c\varphi(\zeta) - \sum_\alpha m_\alpha\lambda_\alpha\right]\frac{\partial\varphi}{\partial\zeta}.$$

The equations of the problem have the form

$$\begin{aligned} \dot{\eta}_{ik} &= \sum_\alpha b_{k\alpha}\eta_{i\alpha} + m_k\varphi(\zeta), \\ \dot{\lambda}_k &= -\sum_\alpha b_{\alpha k}\lambda_\alpha + 2a_k\eta_{ik}, \\ 0 &= \left[2c\varphi(\zeta) - \sum_\alpha m_\alpha\lambda_\alpha\right]\frac{\partial\varphi}{\partial\zeta}. \end{aligned} \tag{2.4}$$

They coincide to within the factor $\partial\varphi/\partial\zeta$ with the equations [(2.1) and (3.3)] of the analogous problem for the open region $N(\xi, \eta_1, \ldots, \eta_n)$ which was formulated in [1].

Therefore, in addition to the solution obtained in [1] for the open region, we should consider the new solution which corresponds to the equation

$$\frac{\partial\varphi}{\partial\zeta} = 0. \tag{2.5}$$

By virtue of the definition given by (2.2), we set

$$\varphi(\zeta) = \pm\bar{\xi} \quad \text{for} \quad |\zeta| \geqslant \zeta^*. \tag{2.6}$$

This latter means that the optimal solution passes over the boundary of the region $\bar{N}(\xi, \eta_1, \ldots, \eta_n)$.

For interior points of the region, the solution found has the form [1]

$$\xi = \sum_{\alpha=1}^{n} p_\alpha\eta_{i\alpha}. \tag{2.7}$$

Thus, the equation of the controller which corresponds to the functional of (1.5) is written as

$$\xi = \begin{cases} \sum_\alpha p_\alpha\eta_{i\alpha} & \text{for} \quad \left|\sum_\alpha p_\alpha\eta_{i\alpha}\right| < \bar{\xi}, \\ +\bar{\xi} & \text{for} \quad \sum_\alpha p_\alpha\eta_{i\alpha} \geqslant \bar{\xi}, \\ -\bar{\xi} & \text{for} \quad \sum_\alpha p_\alpha\eta_{i\alpha} \leqslant -\bar{\xi}. \end{cases} \tag{2.8}$$

Obviously, this equation corresponds to an ideal nonlinear controller with an infinitely large servomotor speed.

The question of the stability of system (1.3), supplied with the controller defined by (2.8), should be considered separately. It is the subject-matter of an independent problem. As was shown in [1], for a deviation $|\Sigma p_\alpha\eta_{i\alpha}| < \bar{\xi}$, stability is guaranteed; for a deviation $|\Sigma p_\alpha\eta_{i\alpha}| \geqslant \bar{\xi}$ a special investigation is required, leading to the construction of the region of attraction, the boundary of which necessarily lies beyond the limits of $|\Sigma p_\alpha\eta_{i\alpha}| = \bar{\xi}$. Various cases can occur here. Thus, in example 1 of paper [1], where $n = 1$, $b_{11} = b$, $m_1 = m$ and, $\eta_1 = \eta$, we had the system of equations

$$\dot{\eta} = b\eta + m\xi,$$

$$\xi = \begin{cases} +\bar{\xi} & \text{for} \quad -\frac{k+b}{m}\eta \geqslant \bar{\xi}, \\ -\frac{k+b}{m}\eta & \text{for} \quad \left|1 - \frac{k+b}{m}\eta\right| < \bar{\xi}, \\ -\bar{\xi} & \text{for} \quad -\frac{k+b}{m}\eta \leqslant -\bar{\xi}, \end{cases} \tag{2.9}$$

where the number $\underline{k}$ is defined by the formula

$$k = +\sqrt{b^2 + \frac{m^2 a}{c}}. \tag{2.10}$$

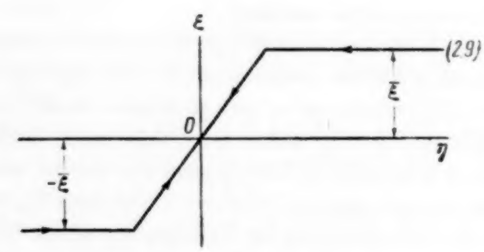

Fig. 2.

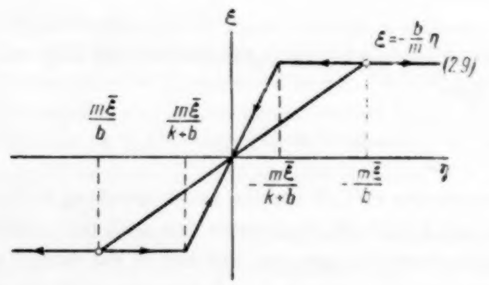

Fig. 3.

Figures 2 and 3 show the system's phase plane for the cases $m, b < 0$ and $m < 0$, $b > 0$. In the first case, the curve $\xi = \xi(\eta)$ (2.9) is the sole integral curve by which the representative point returns to equilibrium for any η_0; in the second case, the system's stability is guaranteed only for $|\eta_0| < -\frac{m\bar{\xi}}{b}$. The features are analogous in the two other cases, in which $m > 0$, $b \gtrless 0$.

It is of interest to note that the choice of sufficiently small weight constants in the function of (1.5) permits an arbitrarily close approach to a relay characteristic of the optimal controller :

$$\xi = \bar{\xi}\,\mathrm{sign}\left(-\frac{k+b}{m}\,\eta\right). \tag{2.11}$$

However, the execution of the limiting transfer itself requires additional discussion.

3. Possible Generalizations of the Functional of (1.5)

We now consider the case of optimizing a functional of the form

$$I(\xi) = \int_0^\infty (V + \dot{\xi}^2)\,dt. \tag{3.1}$$

In accordance with transformation (2.1), we have

$$I = [\varphi(\zeta)] = \int_0^\infty \left[\sum_k a_k \eta_{ik}^2 + c\varphi^2(\zeta) + \left(\frac{\partial\varphi}{\partial\zeta}\dot{\zeta}\right)^2\right] dt.$$

The boundary conditions remain as before, i.e., (1.4). We set

$$H = \Sigma a_k \eta_{ik}^2 + c\varphi^2(\zeta) + \left(\frac{\partial\varphi}{\partial\zeta}\dot{\zeta}\right)^2 + \sum_k \lambda_k\left[\dot{\eta}_{ik} - \sum_\alpha b_{k\alpha}\eta_{i\alpha} - m_k\varphi(\zeta)\right]. \tag{3.2}$$

Since

$$\frac{\partial H}{\partial\dot{\zeta}} = 2\left(\frac{\partial\varphi}{\partial\zeta}\right)^2\dot{\zeta},\quad \frac{\partial H}{\partial\zeta} = \left[2c\varphi(\zeta) + 2\,\frac{\partial^2\varphi}{\partial\zeta^2}\,\dot{\zeta}^2\right]\frac{\partial\varphi}{\partial\zeta} - \sum_k \lambda_k m_k \frac{\partial\varphi}{\partial\zeta}\,,$$

then the equations of the variational problem will have the form

$$\dot{\eta}_{ik} = \sum_\alpha b_{k\alpha}\eta_{i\alpha} + m_k\varphi(\zeta),$$

$$\dot{\lambda}_k = -\sum_\alpha b_{\alpha k}\lambda_\alpha + 2a_k\eta_{ik}, \tag{3.3}$$

$$\frac{d}{dt}\left[2\left(\frac{\partial\varphi}{\partial\zeta}\right)^2\dot{\zeta}\right] = \left[2c\varphi(\zeta) + 2\,\frac{\partial^2\varphi}{\partial\zeta^2}\,\dot{\zeta}^2 - \sum_k \lambda_k m_k\right]\frac{\partial\varphi}{\partial\xi}\,.$$

Consider the last of these equations. After some obvious simplifications, we find that

$$2\left[\frac{\partial\varphi}{\partial\zeta}\ddot{\zeta} + \frac{\partial^2\varphi}{\partial\zeta^2}\dot{\zeta}^2\right]\frac{\partial\varphi}{\partial\zeta} = \left[2c\varphi(\zeta) - \sum_k m_k\lambda_k\right]\frac{\partial\varphi}{\partial\zeta} \tag{3.4}$$

Therefore, (3.3) should be considered for the two cases

$$\frac{\partial\varphi}{\partial\zeta} = 0, \tag{3.5}$$

when the system's motion occurs on the boundary of region $\bar{N}$, and

$$2\left[\frac{\partial\varphi}{\partial\zeta}\ddot{\zeta} + \frac{\partial^2\varphi}{\partial\zeta^2}\dot{\zeta}^2\right] = 2c\varphi(\zeta) - \sum_k m_k\lambda_k, \tag{3.6}$$

when the system's motion takes place on the interior points of region $\bar{N}$. In the first case we have the solution

$$\varphi(\zeta) = \pm\bar{\xi} \quad \text{for} \quad |\zeta| \geqslant \zeta^*. \tag{3.7}$$

In the second case, since $\frac{\partial\varphi}{\partial\zeta}\dot{\zeta} = \frac{d}{dt}\varphi(\zeta)$, Eq. (3.6) reduces to the form

$$2\ddot{\varphi} = 2c\varphi - \sum_k m_k\lambda_k. \tag{3.8}$$

In conjunction with the first 2n equations of (3.3), this determines the controller equation

$$\dot{\xi} = \sum_\alpha p_\alpha\eta_{i\alpha} - r\xi. \tag{3.9}$$

It is essential to note that, inasmuch as Eqs. (3.3) of the variational problem coincide exactly with (2.1) and (3.3) of [1], the equation just found, (3.9), coincides exactly with (4.5) of [1].

On the $\xi, \eta_1, \ldots, \eta_n$ phase space we sketch the two hyperplanes

$$\sum p_\alpha\eta_{i\alpha} = \pm r\bar{\xi}. \tag{3.10}$$

Then, the controller equation being sought will be written as

$$\dot{\xi} = \begin{cases} +\bar{\xi} & \text{for} \quad \dfrac{\Sigma p_\alpha\eta_\alpha}{r} \geqslant \bar{\xi}, \\ \Sigma p_\alpha\eta_{i\alpha} - r\xi & \text{for} \quad \left|\dfrac{\Sigma p_\alpha\eta_\alpha}{r}\right| < \bar{\xi}, \\ -\bar{\xi} & \text{for} \quad \dfrac{\Sigma p_\alpha\eta_\alpha}{r} \leqslant -\bar{\xi}. \end{cases} \tag{3.11}$$

4. The Weierstrass-Erdmann Conditions

To complete the investigation of the necessary conditions for the existence of an optimal solution of the analytic design problem, one should convince oneself that the Weierstrass-Erdmann conditions hold at the nodal points of function ξ.

In the case when one is considering a functional of the form of (3.1), these conditions reduce to the holding of the following equalities:

$$\frac{\partial H^+}{\partial \dot{\eta}_k} = \frac{\partial H^-}{\partial \dot{\eta}_k}, \qquad \frac{\partial H^+}{\partial \dot{\zeta}} = \frac{\partial H^-}{\partial \dot{\zeta}} \qquad (k = 1, \ldots, n), \tag{4.1}$$

$$H^+ - \sum_k \left(\frac{\partial H}{\partial \dot{\eta}_k} \dot{\eta}_k\right)^+ - \left(\frac{\partial H}{\partial \dot{\zeta}} \dot{\zeta}\right)^+ = H^- - \sum_k \left(\frac{\partial H}{\partial \dot{\eta}_k} \dot{\eta}\right)^- - \left(\frac{\partial H}{\partial \dot{\zeta}} \dot{\zeta}\right)^-. \tag{4.2}$$

Here, H is the function in (2.3) or in (3.2), and the "+" and "−" signs denote that the computation of the corresponding functions is carried out to the left or to the right of the nodal points of function ξ.

One easily convinces himself that, if the first $\underline{n}$ conditions of (4.1) hold, which reduce to the equations

$$\lambda_k^+ = \lambda_k^- \qquad (k = 1, \ldots, n), \tag{4.3}$$

then the two remaining conditions are also satisfied for any functions $\xi, \eta_1, \ldots, \eta_n$ of class C. As for (4.3), it can always be satisfied, as is easily verified, if the optimal system is stable.

Thus, in the example considered in Section 2, the solutions for the optimal controller in the case when $|\xi| \geqslant \bar{\xi}$ have the form

$$\begin{aligned} \eta &= c_1 e^{bt} \mp \frac{m\bar{\xi}}{b}, \\ \lambda &= c_2 e^{-bt} \mp \frac{2am}{b^2}\bar{\xi} + \frac{ac_1}{b} e^{bt}, \end{aligned} \tag{4.4}$$

and, in the case when $|\xi| < \bar{\xi}$

$$\begin{aligned} \eta &= \bar{c}_1 e^{-kt} + \bar{c}_2 e^{kt}, \\ \lambda &= \frac{2c}{m^2} [-(k+b)\bar{c}_1 e^{-kt} + (k-b)\bar{c}_2 e^{kt}], \\ \xi &= \frac{1}{m} [-(k+b)\bar{c}_1 e^{-kt} + (k-b)\bar{c}_2 e^{kt}]. \end{aligned} \tag{4.5}$$

From the conditions on the left end (for t = 0)

$$\eta_0 = c_1 \mp \frac{m\bar{\xi}}{b}; \qquad \xi_0 = \pm \bar{\xi}. \tag{4.6}$$

From the conditions on the right end (for t = ∞),

$$\bar{c}_2 = 0. \tag{4.7}$$

The continuity conditions at the nodal points are

$$\bar{c}_1 = c_1 e^{bt_*} \mp \frac{m\bar{\xi}}{b}; \quad -\frac{k+b}{m}\bar{c}_1 = \pm \bar{\xi}. \tag{4.8}$$

The single condition of (4.3) has the form

$$c_2 e^{-bt_*} \mp \frac{2am\bar{\xi}}{b^2} + \frac{ac_1}{b} e^{bt_*} = -\frac{2c}{m^2}(k+b)\bar{c}_1. \tag{4.9}$$

The last equation of (4.1) and (4.2) reduce to identities.

Equations (4.6), (4.7), and (4.9) define the constants c_1, $\bar{c}_7$, and c_2; (4.8) define c_1 and the time t_*, corresponding to the nodal point of function ξ. We, hence, find that

$$\pm \frac{k}{b} \frac{m\bar{\xi}}{k+b} = \left(\eta_0 \pm \frac{m\bar{\xi}}{b}\right) e^{bt_*}. \tag{4.10}$$

It is easily established that, for m < 0 and b < 0, (4.10) is solvable for $t_* > 0$ for any $|\eta_0| \geqslant \left|-\frac{m\bar{\xi}}{k+b}\right|$; if this latter inequality does not hold, the system does not go beyond the boundary $|\xi| = \bar{\xi}$ and no nodal point exists. In the case when m < 0 and b > 0, the equation has a solution with respect to $t_* > 0$ for any η_0 which satisfy the inequality

$$\left|-\frac{m\bar{\xi}}{k+b}\right| \leqslant |\eta_0| \leqslant \left|-\frac{m\bar{\xi}}{b}\right|$$

This is again the case shown on Figs. 2 and 3. The analogous features also occur for the other combinations of signs of $\underline{m}$ and $\underline{b}$.

If the functional of (3.1) is used as the criterion of system optimality, then the Weierstrass-Erdmann conditions also are met. Indeed, the solution of the system for $|\xi| = \bar{\xi}$ is as follows:

$$\begin{aligned} \eta &= c_1 e^{bt} \mp \frac{m\bar{\xi}}{b}, \quad \xi = \bar{\xi}, \\ \lambda &= c_2 e^{-bt} \mp \frac{2am\bar{\xi}}{b^2} + \frac{ac_1}{b} e^{bt}. \end{aligned} \tag{4.11}$$

With the boundary conditions at infinity taken into account, the solution of the system for $|\xi| < \bar{\xi}$ is

$$\begin{aligned} \eta &= \bar{c}_1 e^{\mu_1 t} + \bar{c}_2 e^{\mu_2 t}, \\ \xi &= \frac{\mu_1 - b}{m}\bar{c}_1 e^{\mu_1 t} + \frac{\mu_2 - b}{m}\bar{c}_2 e^{\mu_2 t}, \\ \lambda &= -\frac{2a}{\mu_1 + b}\bar{c}_1 e^{\mu_1 t} - \frac{2a}{\mu_2 + b}\bar{c}_2 e^{\mu_2 t}. \end{aligned} \tag{4.12}$$

We find from the initial conditions that

$$\eta_0 = c_1 \mp \frac{m\bar{\xi}}{b}, \quad \xi_0 = \bar{\xi}. \tag{4.13}$$

The continuity conditions for the functions ξ and η at the nodal points are

$$\bar{\xi} = \frac{\mu_1 - b}{m}\bar{c}_1 + \frac{\mu_2 - b}{m}\bar{c}_2, \quad c_1 e^{bt_*} \mp \frac{m\bar{\xi}}{b} = \bar{c}_1 + \bar{c}_2. \tag{4.14}$$

The Weierstrass-Erdmann conditions reduce to the holding of the two equalities

$$\lambda^+ = \lambda^-, \quad \left[\left(\frac{\partial\varphi}{\partial\zeta}\right)^2 \dot{\zeta}\right]^+ = \left[\left(\frac{\partial\varphi}{\partial\zeta}\right)^2 \dot{\zeta}\right]^- \tag{4.15}$$

or, what amounts to the same thing, to the equalities:

$$\begin{gathered} c_2 e^{-bt_*} \mp \frac{2am\bar{\xi}}{b^2} + \frac{ac_1}{b} e^{bt_*} = -\frac{2a}{\mu_1 + b}\bar{c}_1 - \frac{2a}{\mu_2 + b}\bar{c}_2, \\ \mu_1(\mu_1 - b)\bar{c}_1 + \mu_2(\mu_2 - b)\bar{c}_2 = 0. \end{gathered}$$

Equation (4.13) defines the constant c_1; from (4.14) we find that

$$\bar{c}_1 = \frac{1}{\mu_2 - \mu_1}\left[(\mu_2 - b)\,\eta^* - m\bar{\xi}\right],$$
$$\bar{c}_2 = \frac{1}{\mu_2 - \mu_1}\left[\mu\bar{\xi} - (\mu_1 - b)\,\eta^*\right], \tag{4.17}$$

where $\eta^* = \eta(t_*)$. The first equation of (4.16) defines the constant c_2, the second defines the moment of time t_* when the system passes through the nodal point. This last equation, in correspondence with (4.17), takes the form

$$m\bar{\xi}\,(\mu_1 + \mu_2 - b) = (\mu_1 - b)\,(\mu_2 - b)\,\eta^*. \tag{4.18}$$

This last equation means that, at the nodal point, the servomotor speed must vanish, as was also the case with (5.4) of [1].

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OPTIMAL PROCESSES IN SECOND-ORDER PULSE-RELAY SYSTEMS

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The problem of finding the optimal control law in pulse-relay second-order systems is considered.

Let there be given a physical system which is described by the n-dimensional vector $x_i(t)$ and let the position of the representative point in n-dimensional phase space at the initial moment of time be defined by $x_i(0)$. The representative point must be translated to the given point ξ_i. The process will have been completed after time T, and will be optimal if no other process provides for the attainment of ξ_i in a time less than T.

The optimal control problem for systems described by differential equations of the form

$$\frac{dx_i}{dt} = \varphi_i(x_i, u_l, t) \qquad (i = 0, 1, \ldots, n;\ l = 0, 1, \ldots, r) \tag{1}$$

has been widely investigated in a number of works [1, 2].

In this posing of the problem, time is continuous, and one imposes limitations of the following type on the controlling quantities u_l:

$$|u_l| \leqslant N_l. \tag{2}$$

In [3] the analogous problem is solved for systems described by finite difference equations

$$x_i([k+1]h) - x_i(kh) = \Big[\sum_{j=1}^{n} a_{ij}x_j(kh) + \sum_{l=1}^{r} q_{il}u_l(kh) + f_i(kh)\Big]h. \tag{3}$$

In this problem, time is discrete, $t = kh$ $(k = 0, 1, \ldots)$ for an arbitrarily chosen h, and conditions of the following type are imposed on the controlling quantities:

$$|u_l(kh)| \leqslant N. \tag{4}$$

Such a system was investigated in [4].

The system of second-order finite difference equations which we consider in this paper, these equations describing planar motion, has the form

$$\begin{aligned} x([k+1]h) - x(kh) &= y(kh)h, \\ y([k+1]h) - y(kh) &= u(kh)h, \end{aligned} \tag{5}$$

where x, y are the point coordinates on the phase plane and u is the point acceleration (controlling action). The behavior of the points is considered under the limitation

$$|u_k| = \text{const}. \tag{6}$$

In this case (a pulse-relay system), it is required to determine the control law $\{u_k^0\}$ which allows the initial point (x_0, y_0) to be translated to the origin of coordinates of phase space in the least possible number of steps K^0. The time here, as in (3), is discrete but, in contradistinction to (1) and to (3), the values of the controlling quantities are also discrete.

The optimal control problem for a system with (5), which we shall solve in the present work, can be classified as a simple number-theoretical problem. It should be mentioned that problems of this type will continue to arise with ever greater frequency in connection with the use of discrete computer technology in automatic control.

It seems advisable to sketch briefly our plan for finding the optimal control law for such a problem. We first find the optimal control law for initial points of the form $(n^2, 0)$, for which we shall use the results of the work of N. N. Krasovskii [3] (the corresponding expressions for the number of steps K^0 and for the control law $\{u^0\}$ can be easily and immediately obtained). We then construct the optimal control for all the integral points of the x axis, $x > 0$, and we show that the optimal trajectories completely fill up the region of the (x, y) plane defined by the expressions $0 \leq x \leq y(y-1)/2$, $y < 0$. Finally, we shall show that the optimal trajectories of all the other integral points of the (x, y) plane pass through one of the points already considered on the x axis in one of the half planes $x > 0$ or $x < 0$. On the basis of the optimal control law obtained, we shall construct an optimal switching line. At the end of this paper we shall discuss questions related to the nonsinglevaluedness of the control law obtained.

We turn now to the solution of the problem, to be carried out in accordance with the plan just given.

The solution obtained in [3] for a second-order system with (4) renders it possible, for a given h, to choose an optimal value of $|u_k|$ which depends on x_{i0} and ξ_i, but does not depend on the number k of steps, and to determine the unique moment for changing the sign of the controlling action (corresponding to a transition from a type N to a P-trajectory [1]).

The peculiar feature of our problem, when (4) been replaced by (6), is that, generally speaking, for a constant modulus of point acceleration, the sign of this acceleration changes more than once during the control process.

Indeed, we consider a point lying on the $\underline{x}$ axis at a distance x_0 from the origin of coordinates. Let $\underline{x}$ and $\underline{y}$ be changed for integer steps, i.e., $|u_k| = 1$, $h = 1$. If $|u_k| = a$ and $h = b$, where $a, b \neq 1$, this leads only to a corresponding change in the scales of the process. In accordance with [3], when (4) holds, K^0 is defined as the least integer which satisfies the inequality

$$\frac{K^2h^2}{4} \geqslant x_0.$$

Replacement of (4) by (6) entails only the change of the inequality sign to an equality sign, i.e.,

$$\frac{K^2h^2}{4} = x_0. \tag{7}$$

In order that (7) have integer solutions in K, it is necessary that x_0 be a perfect square, i.e., $x_0 = n^2$. For such points of the $\underline{x}$ axis, the optimal control law of [3] remains valid, it taking, in the given case, the form

$$u^0(kh) = -\operatorname{sign}\left[\frac{K^0}{2} - k\right],$$

$$K^0 = 2\sqrt{x_0} = 2n. \tag{8}$$

In the case of (6), for all the remaining points of the $\underline{x}$ axis, it is not possible to reach the origin of coordinates with a single switching of the acceleration's sign.

We limit ourselves initially to a consideration, for such points, of the half-plane $x > 0$, and to the requirement that one attain in the least number of steps some neighborhood of the origin of coordinates consisting, for example, of the two points (0,0) and (1,0). We define the position of the point on the $\underline{x}$ axis by the distance $\underline{p}$:

$$x_0 = n^2 + p,$$

where

$$0 < p \leqslant 2n + 1.$$

Since the chosen neighborhood is characterized by $y_{K^0} = 0$ then, for $y_0 = 0$, we have the equation

$$\sum_{u_i > 0} u_i = -\sum_{u_j < 0} u_j \quad (i, j = 1, 2, \ldots, K^0), \tag{9}$$

i.e., the neighborhood of zero may be reached only in an even number of steps. It can be shown that

$$K^0 = \begin{cases} 2n, & \text{if} \quad p = 1 \\ 2n + 2, & \text{if} \quad p = 2, 3, \ldots, 2n + 1. \end{cases} \tag{10}$$

In fact, for a zero initial velocity, the path taken by the point in 2n steps is maximal for the choice of $u_0(kh)$, according to (8), and is equal to n^2. With this, the point is at the distance $\underline{p}$ from the origin of coordinates, so that, for $p = 1$, the point falls in the neighborhood of the origin of coordinates, and (8) defines its optimal control law. When (9) is taken into account, for $p \neq 1$, the magnitude of K^0 cannot be less than $2n + 2$.

We now assume that the optimal control law has the form:

$$u_k^0 = -\operatorname{sign}\left[\frac{K^0}{2} - k - 1\right] - 2\delta_{kz},$$

$$\delta_{kz} = \begin{cases} 1 & \text{for} \quad z = k \\ 0 & \text{for} \quad z \neq k \quad (k = 1, 2, 3, \ldots, K^0), \end{cases} \tag{11}$$

where K^0 is defined by (10).

Now we show that one can choose a $\underline{z}$ corresponding to an optimal control.* As the end point for an even $p = 2l$, we choose the point $x_{K^0} = 1$; for $p = 2l + 1$, let such a point be $x_{K^0} = 0$ ($l = 1, 2, \ldots$). Then $\underline{z}$ must be so chosen that, for any parity of $p (p \neq 1)$, the following equation holds:

$$S_{K^0} + E\left[\frac{p}{2}\right] - E\left[\frac{p-1}{2}\right] = n^2 + p,$$

where S_{K^0} is the path taken by the system during the control process in K_0 steps; $E(\alpha)$ denotes the integral part of α. At the same time, by adding the segments of path taken by the system for the controlling actions of (11), we obtain

$$S_{K^0} = n^2 + 2[2n - 1 - (z - 1)] - 1.$$

By comparing the two last formulas, we obtain the expression for $\underline{z}$:

$$z = K^0 - E\left[\frac{p+1}{2}\right]. \tag{12}$$

Consequently, if $\underline{z}$ is defined by (12), then (11) defines the optimal control law.

The optimal law just obtained is easily extended to any integral point of the phase half plane $x > 0$. In fact, the initial points $x_0 = \operatorname{sign} y_0$ for $x_0 \geq y_0(y_0-1)/2$ (the boundary of this region is the N-trajectory which terminates in the origin of coordinates, i.e., $x_0 = y_0(y_0-1)/2$) lie on one of the optimal trajectories considered. One easily convinces himself that at least one optimal trajectory passes through each point of this region. This follows from the fact that the optimal control law, up until the first switchings (i.e., for $k \leq K^0/2-1$), does not depend on $\underline{p}$. The points $\operatorname{sign} x_0 = \operatorname{sign} y_0$ and $\operatorname{sign} x_0 = -\operatorname{sign} y_0$, with the condition that $x_0 < y_0(y_0-1)/2$, are taken over into the case just considered after y_0 steps $u_k = -\operatorname{sign} x_0$ in the right-and left-hand planes respectively. It is easily seen that the "initial translation" of points moving along P- or N-trajectories are also optimal. If necessary, the optimal control law can be written for arbitrary initial conditions. For example, for the case of initial points defined by the conditions

$$\operatorname{sign} x_0 = -\operatorname{sign} y_0, \quad x > {}^1/_2\, y_0(y_0 - 1) \quad (p \neq 1),$$

$$K^0 = 2(n + 1) - |y_0|,$$

$$u_k^0 = -\operatorname{sign}[n - |y_0| - k] - 2\delta_{kz},$$

* We adopt the convention that sign 0 = 1.

we have

$$z = K^0 - E\left[\frac{p+1}{2}\right] - |y_0|,$$

$$p = x_0 + {}^1/_2\, y_0(y_0+1) - n^2.$$

The control law just found corresponds to a translation of points to the neighborhood of zero, but it is easily verified that if K^0 remains invariant, and for any number of switchings, if (9) is met then the parity of S_k is unchanged. Therefore, the law that translates the point directly to the origin of coordinates must be supplemented by two steps (when $u_{K^0+1} = -u_{K^0+2} = 1$) to all the trajectories which terminate in the point (1,0). Then, the optimal control law, in the form of (11 and (12), will be valid for all points and, for all p, we shall have $K^* = 2n+2$.

The control law in the form of (11) and (12) allows us to find the optimal switching line.

We consider the phase half-plane $x > 0$ and the half-line $y = y^*$ on it. It is obvious that the optimal stimulus for all points with integral coefficients on the half-line $y = y^* > 0$ will be $u^0 = -\text{sign } x = -1$. Now let $y^* < 0$ and $v = |y^*|$ (Fig. 1). The line under consideration intersects the N-trajectory $x = v(v-1)/2$ at point $\{a[v(v-1)/2, y^*]\}$. Then, for all points with coordinates $0 \le x \le v(v-1)/2$; $y = y^*$, we have that $u^0 = \text{sign } x = +1$. It

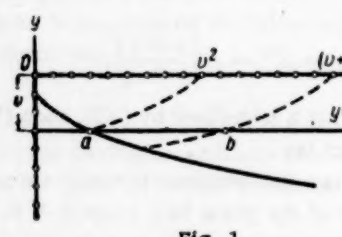

Fig. 1.

remains to consider the points $x > v(v-1)/2$, $y = y^* > 0$. We select an interval of the x axis which contains $2v+1$ points $v^2 \le x_0 < (v+1)^2$. For all points, according to (10), $K^0 = 2v^2 + 2$ (in the sense of a translation to the neighborhood of the origin of coordinates) and (11), it is required that $u^0 = \text{sign } x = 1$, since a point of such a segment falls on the line after v steps. For points on the x axis such that $x_0 \ge (v+1)^2$, the first change of sign occurs on the step whose ordinal number is known to be greater than v, and for these points $u^0 = -\text{sign } x$. Thus, for $y_0 = -v$, the optimal control is these points $u^0 = -\text{sign } x$. Thus, for $y_0 = -v$, the optimal control is

$$u^0 = \begin{cases} -\text{sign } x_0, & \text{if } \quad x \geqslant b, \\ +\text{sign } x_0. & \text{if } \quad x < b. \end{cases}$$

The abscissa of point b is obviously equal to $v(v-1)/2 + (2v+1) = (v+1)(v+2)/2$. Since the argument did not depend on the choice of v, one may assert that the expression just found defines the optimal switching line on the half-plane $x > 0$.

By carrying out an analogous argument for the half-plane $x < 0$, we can obtain the complete expression for the optimal switching line:

$$u^0 = \begin{cases} -1, & \text{if} \quad x \geqslant f(v, x), \\ +1, & \text{if} \quad x < f(v, x), \end{cases} \tag{13a}$$

where $v = |y|$.

$$f(v, x) = \begin{cases} f_1(v) = \dfrac{(v+1)(v+2)}{2}, & \text{if} \quad x > 0, \\ f_2(v) = \dfrac{v(v+3)}{2}, & \text{if} \quad x < 0. \end{cases} \tag{13b}$$

In the half-plane $x < 0$, to retain the "$\ge$" and "$<$" signs unchanged, the switching line must be translated by unity toward the y axis.

The switching line just obtained is optimal in the sense that the representative point is translated either to the origin of coordinates or to a neighborhood of it.

The optimal control law, which we have found in the form of (11) and (12), is not unique. We now show that it is possible to define other forms of the control law. We consider the control of points on an expanse of $2n+2$ steps. A control for which the sign of the stimulus is constant (for example, a negative stimulus) on the first $n+1$ steps ($i = 1, 2, \ldots, n+1$) and, for $i = n+2, \ldots, 2n+2$, the sign is reversed, we shall call a "canonical" control. With canonical control, there are $(n+1)^2$ points on the path. Since, according to (9), the numbers of steps with "+" and with "−" signs are equal, then all possible controls can be obtained from the canonical one by permutations of signs pairwise between both sign-invariant portions. If, for example, we change the signs of the two terms with ordinal numbers m_1 and m_2

$$1 < m_1 \leqslant n+1,\ n+2 \leqslant m_2 \leqslant 2n+2,$$

then the points passed through on the path will be fewer than with the canonical control, and will be defined by the expression

$$S_{2n+2} = (n+1)^2 - 2(m_2 - m_1) \tag{14}$$

If the initial point is defined by the conditions $x = n^2 + p$, $y = 0$, then we can obtain the expression for its "optimal" control laws by equating $S_{2n+2} = n^2 + p$. From this, we obtain the relationship for m_1 and m_2:

$$m_2 - m_1 = \frac{2n-p}{2}. \tag{15}$$

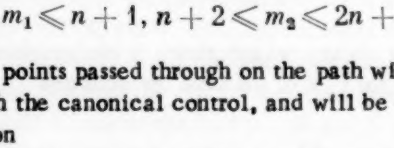

Fig. 2.

In this work, we fixed the value of $m_1 = n+1$, and m_2 varied as a function of p. Such an approach made it possible to obtain the optimal switching line.

A typical form of controlling stimulus is illustrated on Fig. 2. Here, $x_0 = -19$ and $y_0 = 0$.

Figure 3 shows the optimal trajectory for the point ($x_0 = 12$, $y_0 = 0$) which corresponds to n = 3 and p = 3. Trajectory 1 is formed in accordance with optimal switching line 3, and trajectory 2 (dashed lines) is obtained from the control of (15) for $m_1 = n$.

Fig. 3.

Figure 4 shows one of the simplest variants of a realization of an optimal control system using switching lines. The system's action is based on the determination of the mutual positioning of the representative point and the switching line in the half-plane where $x > 0$.

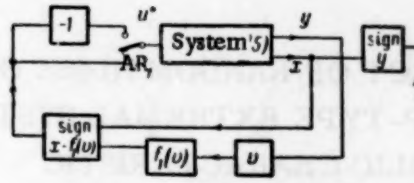

Fig. 4.

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THE EFFECT OF RANDOM NOISE ON THE STEADY-STATE OPERATION OF A STEP-TYPE EXTREMAL SYSTEM FOR AN OBJECT WITH A PARABOLIC CHARACTERISTIC

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The optimal parameters of the controlling portion of the extremal system under consideration are calculated. These parameters must be determined from estimates of the deviations from the minimum characteristics in the steady state.

It is shown that, in the particular case of uniformly distributed noise increments, it is most advantageous to operate with small steps, to be determined below.

1. Posing of the Problem

In this paper we consider the case of a step-type extremal system whose object has a parabolic characteristic $y = x^2$. The system's task is the automatic search for, and maintenance of, the minimum of the characteristic. The following posing of the problem, as well as the notation, are the same as in [1] of A. A. Fel'dbaum. All quantities are considered at discrete moments of time $t = nT$ ($n = 0, 1, 2, \ldots$), where T is the time between two successive working steps, or the duration of one cycle. Figure 1 is a block schematic of the extremal system under consideration; O is the object of control, its input and output being related by the function

$$y[n] = x^2[n]. \tag{1}$$

Random noise z[n] acts at the object's output. The noise increments $v = z[n+1] - z[n]$ are assumed to be statistically independent quantities with probability density $\rho(v)$ and zero mean value. The noise z[n] can be identified, for example, with the random noise in the measuring block of an actual system. Slow shifts of the characteristic, or input noise of the object, will not be considered here. The action of noise retards the automatic search for the minimum of the object's characteristic, where the minimum itself is determined with a certain error.

The actual object output, which can be measured, is the sum of the output y[n] and the random noise, i.e., the quantity

$$w[n] = y[n] + z[n]. \tag{2}$$

The quantity w[n] is applied to the input of the controlling portion of the system C.

The search for a minimum is implemented by means of successive cycles; each cycle includes a test to determine the direction of motion toward a minimum, and a working

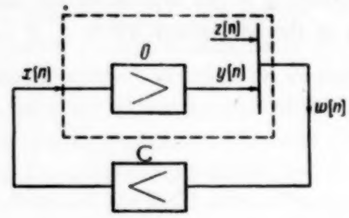

Fig. 1

operation which implements this motion. The controlling portion, with its testing motions, disturbs the object's regimen. In the system under consideration, at the beginning of each half cycle, the object's input x [n] receives a stepwise increment of $(\Delta x)_1 = a$ (at the beginning of the cycle) and $(\Delta x)_2 = -a$ (in the middle of the cycle). At the object's output, at the end of each half cycle, one measures the corresponding disturbances $w_1 = y_1 + z_1$ and $w_2 = y_2 + z_2$ and compares them. Thus, one determines the direction of motion toward a minimum, or the sign of the working step (the magnitude of the working step, as of the test step, is a):

$$\begin{aligned} \Delta x[n] &= -a \operatorname{sign} \Delta w[n], \\ \Delta w[n] &= w_1 - w_2. \end{aligned} \tag{3}$$

It is possible with this to make a false step, on moving away from the minimum, since the quantity w is made up of the true output of the object plus noise. If one know the distribution law of the noise increments, $\rho(v)$, then the probability of a false step from the point x equals

$$p(x) = P\{v > \Delta y\},$$

where $\Delta y = y_1 - y_2$, $v = z_1 - z_2$, or

$$p(x) = \int_{-\infty}^{-|\Delta y|} \rho(v)\, dv. \tag{4}$$

The probability of a correct step (one directed toward the minimum) from point x equals

$$q(x) = 1 - p(x).$$

Possible neighboring points $\underline{x}$ are at a distance equal to the working step $\underline{a}$ from each other. We renumber the possible discrete states of the system, assigning the subscript zero to the state which corresponds to the minimum of the characteristic, i.e.,

$$x = ia \qquad (i = -\infty, \ldots, -1, 0, 1, \ldots, \infty). \tag{5}$$

We introduce the notation: $p_i = p(x_i)$ is the probability of a false step from x_i, $q_i = q(x_i)$ is the probability of a correct step from x_i.

If p_{jk} is the transition probability from point x_j to point x_k then, in our case,

$$\begin{gathered} p_{k,\,k+1} = \begin{cases} p_k & (k > 0), \\ q_k & (k < 0), \end{cases} \\ p_{k,\,k-1} = \begin{cases} q_k & (k > 0), \\ p_k & (k < 0), \end{cases} \\ p_{k,\,l} = 0 \qquad (l \neq k-1,\ k+1). \end{gathered} \tag{6}$$

Thus, the process of seeking a minimum can be considered as a discrete Markov process.

We denote by the "weight" of state x_i the mean value of the object's outputs for the two tests, using the notation $(\overline{\Delta y})_i$ for this quantity. In correspondence with (1) and (5),

$$(\overline{\Delta y})_i = a^2 (i^2 + 1) \qquad (i = 0, \pm 1, \pm 2, \ldots). \tag{7}$$

The difference of the values of $\underline{y}$ in the test, $\Delta y = y_1 = y_2$, is defined as

$$(\Delta y)_i = 4ia^2 \qquad (i = 0, \pm 1, \pm 2, \ldots). \tag{8}$$

The probabilities of false and of correct steps from state x_i are

$$p_i = \int_{-\infty}^{-4|i|a^2} \rho(v)\,dv, \qquad q_i = 1 - p_i. \tag{9}$$

The steps from the minimum, both at the point $x_{+1} = a$ and at the point $x_{-1} = -a$ are false, with a probability of $p_0 = 1/2$.

For convenience and clarity of the exposition, we can construct an equivalent scheme of the discrete Markov process [2]. Such a scheme is shown in Fig. 2. The points on it represent the possible states of the system. The arrows show the transitions from state to state. The probabilities of these transitions are shown. Z is the operator for translation in time, showing that the transition is carried out during a working cycle.

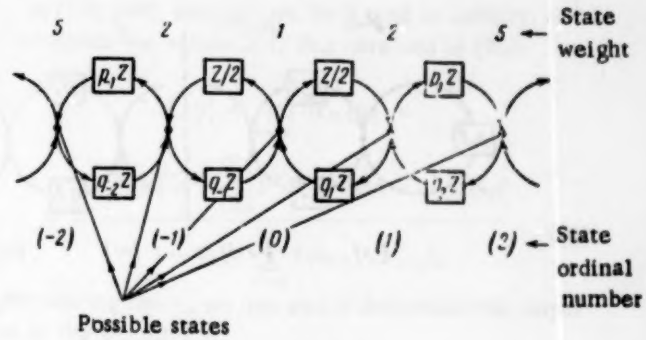

Fig. 2

We denote by $P_i[n]$ the probability that the system is in state x_i at time $t_n = nT$. Then, the mathematical expectation of the output quantity is

$$M_n[y] = \sum_{i=-\infty}^{\infty} (\overline{\Delta y})_i P_i[n] = a^2 \sum_{i=-\infty}^{\infty} (i^2 + 1) P_i[n]. \tag{10}$$

It is necessary to determine $\lim_{n\to\infty} M_n[y]$, if this limit exists, and also to find the $\underline{a}$ for which this limit is a minimum.

The case of an extremal control system with the piecewise-linear characteristic $y = |x|$ was devoted in work [1]. It was shown there that, in the steady state, the minimum value of the mean error in $\underline{y}$ corresponds to a step of size $a \approx 0$ for normally or uniformly distributed output noise.

2. Determination of the Mathematical Expectation and the Dispersion of the Object's Output Variable in the Steady State

We write the equations for the probabilities of the system's states:

$$\begin{aligned} &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ P_{-2}[n] &= P_{-3}[n-1]\,q_3 + P_{-1}[n-1]\,p_1, \\ P_{-1}[n] &= P_{-2}[n-1]\,q_2 + P_0[n-1]\,\frac{1}{2}, \\ P_0[n] &= P_{-1}[n-1]\,q_1 + P_1[n-1]\,q_1, \\ P_1[n] &= P_2[n-1]\,q_2 + P_0[n-1]\,\frac{1}{2}, \\ P_2[n] &= P_3[n-1]\,q_3 + P_1[n-1]\,p_1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{11}$$

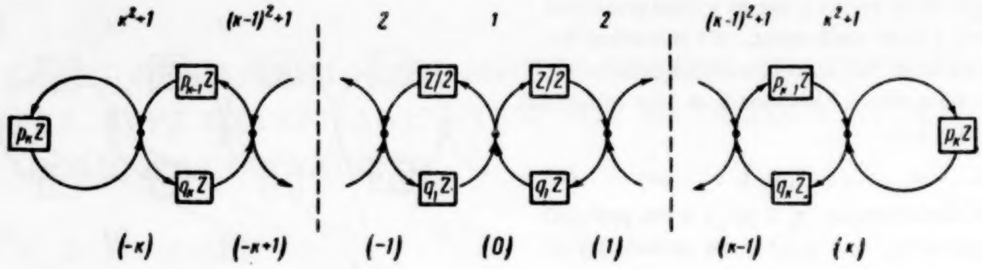

Fig. 3

We have taken into account here that, in correspondence with (9), $p_i = p_{-i}$ and $q_i = q_{-i}$.

We now consider the finite system of 2k + 1 states, shown on Fig. 3. The equations of such system are written in the form

$$
\begin{aligned}
P_{-k}[n] &= P_{-k+1}[n-1]\,p_{k-1} + P_{-k}[n-1]\,p_k,\\
P_{-k+1}[n] &- P_{-k+2}[n-1]\,p_{k-2} + P_{-k}[n-1]\,q_k,\\
&\cdots\cdots\cdots\cdots\cdots\\
P_{-2}[n] &= P_{-1}[n-1]\,p_1 + P_{-3}[n-1]\,q_3,\\
P_{-1}[n] &= P_0[n-1]\frac{1}{2} + P_{-2}[n-1]\,q_2,\\
P_0[n] &= P_{-1}[n-1]\,q_1 + P_1[n-1]\,q_1,\\
P_1[n] &= P_0[n-1]\frac{1}{2} + P_2[n-1]\,q_2,\\
P_2[n] &= P_1[n-1]\,p_1 + P_3[n-1]\,q_3,\\
&\cdots\cdots\cdots\cdots\cdots\\
F_{k-1}[n] &= P_{k-2}[n-1]\,p_{k-2} + P_k[n-1]\,q_k,\\
P\ [n] &= P_{k-1}[n-1]\,p_{k-1} + P_k[n-1]\,p\,.
\end{aligned} \tag{12}
$$

We consider the limiting system for $n \to \infty$, assuming that the $\lim\limits_{n\to\infty} P_i[n] = r_i$ exists. In this case,

$$
\begin{aligned}
r_{-k} &= r_{-k+1}p_{k-1} + r_{-k}p_k,\\
r_{-k+1} &= r_{-k+2}p_{k-2} + r_{-k}q_k,\\
&\cdots\cdots\cdots\\
r_{-2} &= r_{-1}p_1 + r_{-3}q_3,\\
r_{-1} &= r_0\frac{1}{2} + r_{-2}q_2,\\
r_0 &= r_{-1}q_1 + r_1q_1,\\
r_1 &= r_0\frac{1}{2} + r_2q_2,\\
r_2 &= r_1p_1 + r_3q_3,\\
&\cdots\cdots\cdots\\
r_{k-1} &= r_{k-2}p_{k-2} + r_kq_k,\\
r_k &= r_{k-1}p_{k-1} + r_kp_k.
\end{aligned} \tag{13}
$$

It is obvious that

$$\sum_{i=-k}^{k} r_i = 1. \tag{14}$$

We always have 2k + 2 equations in 2k + 1 unknowns. We consider the system of the first 2k equations of (13) and (14), and we determine the r_i which satisfy this system. It will easily be shown that they also satisfy the last equation of (13).

If we also take into account that, by virtue of symmetry, $r_i = r_{-i}$, we shall consequently consider the system

$$
\begin{aligned}
r_0 &= 2r_1q_1,\\
r_1 &= r_0\frac{1}{2} + r_2q_2,\\
r_2 &= r_1p_1 + r_3q_3,\\
&\cdots\cdots\cdots\\
r_{k-1} &= r_{k-2}p_{k-2} + r_kq_k\\
r_0 &+ 2\sum_{i=1}^{k} r_i = 1.
\end{aligned} \tag{15}
$$

We shall successively express all the r_i in terms of r_1:

$$
\begin{aligned}
r_0 &= 2q_1r_1,\\
r_2 &= r_1\frac{p_1}{1}\frac{1}{q_2},\\
r_3 &= r_1\frac{p_1}{1}\frac{p_2}{q_2}\frac{1}{q_3}.\\
&\cdots\cdots\cdots
\end{aligned}
$$

It is easily shown that, in general,

$$r_i = r_1p_1\lambda_2\lambda_3\ldots\lambda_{i-1}\zeta_i \qquad (2 \leqslant i \leqslant k), \tag{16}$$

where

$$\lambda_m = \frac{p_m}{q_m}, \quad \zeta_m = \frac{1}{q_m} \quad (m = 1, 2, \ldots, k). \tag{17}$$

By substituting, in the last equation of (15), the expression for r_0 from the first equation and the expression for r_i from (16), we obtain

$$r_1 = \frac{1}{2\left(1 + q_1 + \sum\limits_{i=2}^{k} p_1\lambda_2\lambda_3\ldots\lambda_{i-1}\zeta_i\right)}. \tag{18}$$

The value of r_0 is defined in terms of r_1 from the first equation of (15)

$$r_0 = 2r_1q_1. \tag{19}$$

It is easily verified that the thus determined r_i satisfy the last equation of (13), i.e., the one we didn't consider. Thus, the state probabilities in the steady state are completely given by (16)-(19).

We now turn back to (10). We need to determine the magnitude of the mathematical expectation in the steady state, i.e.,

$$\lim_{n\to\infty} M_n[y] = a^2 \sum_{i=-\infty}^{\infty} (i^2+1) r_i. \qquad (20)$$

In (16), (18), and (19) we let $\underline{k}$ tend to infinity, and we substitute the values of r_i thus obtained in (20):

$$M[y] = \lim_{n\to\infty} M_n[y] =$$

$$= a^2 \frac{1 + 2\zeta_1 + \sum_{i=2}^{\infty} (i^2+1)\lambda_1\lambda_2\ldots\lambda_{i-1}\zeta_i}{1+\zeta_1+\sum_{i=2}^{\infty}\lambda_1\lambda_2\ldots\lambda_{i-1}\zeta_i}. \qquad (21)$$

Knowing the r_i, we can easily determine the dispersion of the quantity $\underline{y}$:

$$D[y] = \sum_{i=-\infty}^{\infty} (\overline{\Delta y})_i^2 r_i - M^2[y] =$$

$$= a^4 \left\{ \frac{\zeta_1 + \sum_{i=2}^{\infty} i^2\lambda_1\lambda_2\ldots\lambda_{i-1}\zeta_i}{1+\zeta_1+\sum_{i=2}^{\infty}\lambda_1\lambda_2\ldots\lambda_{i-1}\zeta_i} + 2\frac{M[y]}{a^2} - 1 \right\} - M^2[y]. \qquad (22)$$

3. The Case of a Uniform Distribution Law for $\rho(v)$

In this particular case, the computations of M[y] and D[y] by (21) and (22) are easily reduced to numerical results. Let (cf., Fig. 4, a)

$$\rho(v) = \begin{cases} \frac{1}{2\eta} & (|v| \leqslant \eta), \\ 0 & (|v| > \eta). \end{cases} \qquad (23)$$

In correspondence with (9), the probabilities of false and correct steps are found from the formulas

$$p_i = \begin{cases} \frac{1}{2} - i\frac{4a^2}{2\eta} & \left(i < \frac{\eta}{4a^2}\right), \\ 0 & \left(i > \frac{\eta}{4a^2}\right); \end{cases} \qquad (24)$$

$$q_i = \begin{cases} \frac{1}{2} + i\frac{4a^2}{2\eta} & \left(i < \frac{\eta}{4a^2}\right), \\ 1 & \left(i > \frac{\eta}{4a^2}\right) \quad (i = 0, 1, \ldots). \end{cases}$$

We denote by $\underline{k}$ the integral part of the expression $(\eta - 4a^2)/4a^2$, $\eta = 4a^2k + \beta$, $\gamma = \beta/4a^2$ (Fig. 4,b)

Then, from (21),

$$M[y] = a^2 \frac{1 + 2\zeta_1 + \sum_{i=2}^{k+1} (i^2+1)\lambda_1\lambda_2\ldots\lambda_{i-1}\zeta_i}{1+\zeta_1+\sum_{i=2}^{k+1}\lambda_1\lambda_2\ldots\lambda_{i-1}\zeta_i},$$

where λ_m and ζ_m are determined from (17), with account being taken of (24).

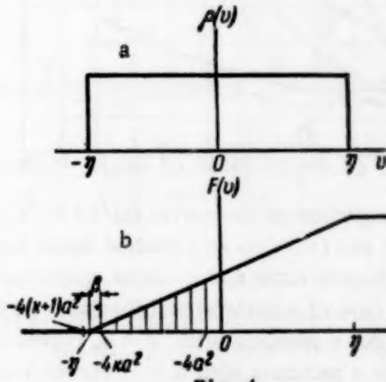

Fig. 4

To simplify the computations, we shall consider, not all $\underline{a}$, but only those for which

$$\frac{\eta}{4a^2} = m, \qquad (25)$$

where $\underline{m}$ is an integer.

We easily obtain from (20), (23), and (24) (cf., the appendix)

$$M[y] = \begin{cases} \frac{\eta}{8} + a^2 & \left(\frac{\eta}{4a^2} > 1\right), \\ 1.5a^2 & \left(\frac{\eta}{4a^2} < 1\right). \end{cases} \qquad (26)$$

Here, since M[y] is a continuous function of $\underline{a}$, $\eta/4a^2$ can take any value, either integral or fractional.

We compute the dispersion D[y] analogously (cf., the appendix)

$$D[y] = \begin{cases} \frac{\eta}{16}\left(\frac{\eta}{2} - a^2\right) & \left(\frac{\eta}{4a^2} > 1\right), \\ \frac{a^2}{4} & \left(\frac{\eta}{4a^2} < 1\right), \end{cases} \qquad (27)$$

or we compute the mean-square deviation

$$\sigma[y] = \begin{cases} \frac{1}{4}\sqrt{\eta\left(\frac{\eta}{2} - a^2\right)} & \left(\frac{\eta}{4a^2} > 1\right), \\ \frac{a^2}{2} & \left(\frac{\eta}{4a^2} \leqslant 1\right). \end{cases} \quad (28)$$

The dependence of M[y] and σ[y] on a for several different values of η is given on Fig. 5. For a comparison on the same graph, we give the dependence of the mathematical expectation and the mean-square dispersion on the size of the working step for the case of the piecewise-linear characteristic y = | x |, considered in [1], for the same noise distribution law. The corresponding computations are given in the appendix.

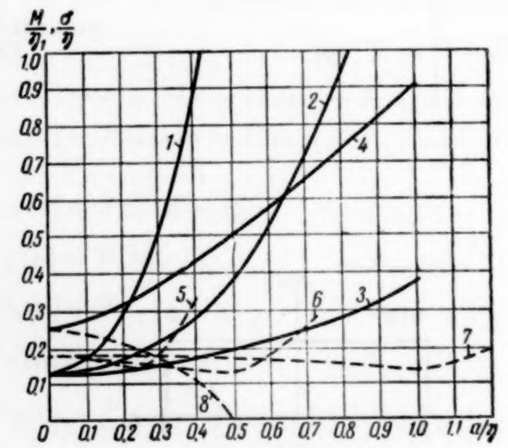

Fig. 5. Comparison of the curves $(M/\eta)(a/\eta)$ (solid lines) and $(\sigma/\eta)(a/\eta)$ (dashed lines) for parabolic characteristic and modulus characteristic in the case of a uniform distribution: curves 1 and 5 are for a parabola with $\eta = 4$, curves 2 and 6 are for a parabola with $\eta = 1$, curves 3 and 7 are for a parabola with $\eta = 0.25$, curves 4 and 8 are for the modulus.

It is clear from the graphs that, for the characteristic $y = x^2$, the optimal (from the point of view of simultaneous smallness of the mathematical expectation and the dispersion) working step a is close to a = 0. A small increase of the step decreases the mean-square deviation insignificantly, but increases the mathematical expectation of the output. In the modulus case, y = | x | , it is better to make the step larger than for the relationship $y = x^2$ since, for the modulus, an increase in the step (for small values of the step a) gives an insignificant increase of M[y] accompanied by a decrease of the dispersion.

In conclusion, the author wishes to thank A. A. Fel'dbaum for posing the given problem and for his useful advice. Thanks also to S. Ya. Raevskii for discussing the results obtained.

APPENDIX

1. Determination of the Mathematical Expectation and the Dispersion of the Output Variable for Uniformly Distributed Noise

By substituting (25) in (24) we get

$$p_i = \begin{cases} \frac{1}{2} - \frac{i}{2m} & (i < m), \\ 0 & (i \geqslant m); \end{cases} \qquad q_i = \begin{cases} \frac{1}{2} + \frac{1}{2m} & (i < m), \\ 1 & (i \geqslant m). \end{cases} \quad \text{(I)}$$

We find from (21), for m > 1,

$$M[y] = a^2 \frac{1 + 2\zeta_1 + \sum_{i=2}^{m} (i^2+1)\lambda_1\lambda_2\ldots\lambda_{i-1}\zeta_i}{1 + \zeta_1 + \sum_{i=2}^{m} \lambda_1\lambda_2\ldots\lambda_{i-1}\zeta_i},$$

where λ_l and ζ_l are determined from (17) with account being taken of (I).

After some transformations, we obtain the expression

$$M[y] = a^2 \frac{2\sum_{i=0}^{m}(i^2+1)\frac{m!^2}{(m-i)!(m+i)!} - 1}{2\sum_{i=0}^{m}\frac{m!^2}{(m-i)!(m+i)!} - 1}.$$

We let $A(m) = \sum_{i=0}^{m} \frac{1}{(m-i)!(m+i)!}$. Then, since

$$\sum_{j=0}^{2m} \frac{(2m)!}{j!(2m-j)!} = 2^{2m} \text{ then } A(m) = \frac{2^{2m-1}}{(2m)!} + \frac{1}{2m!^2},$$

and the denominator of M[y] is written as

$$2m!^2 \frac{1}{2}\left[\frac{2^{2m}}{(2m)!} + \frac{1}{m!^2}\right] - 1 = \frac{2^{2m}m!^2}{(2m)!}.$$

We now consider the numerator of M[y]

$$2\sum_{i=0}^{m}(i^2+1)\frac{m!^2}{(m-i)!(m+i)!} - 1 =$$

$$= 2m!^2(m^2+1)\sum_{i=0}\frac{1}{(m-i)!(m+i)!} -$$

$$- 2m!^2\sum_{i=0}^{m}\frac{m^2-i^2}{(m-i)!(m+i)!} - 1 = \frac{2^{2m-1}m!^2}{(2m)!}(m+2).$$

Whereupon $M[y] = a^2(m+2)/2$ for m > 1.

By taking into account that $m = \eta/4a^2$, we obtain finally,

$$M[y] = \frac{\eta}{8} + a^2 \quad \text{for} \quad \frac{\eta}{4a^2} > 1.$$

For m = 1, we easily get from (21) and (24) that $M[y] = 1.5a^2$; the same result is also obtained for non-integer m less than unity.

The dispersion is computed from (22), with account being taken of (24), (25), and (26).

$$D[y] = a^4 \left\{ \frac{\zeta_1 + \sum_{i=2}^{m} i^4 \lambda_1 \lambda_2 \ldots \lambda_{i-1} \zeta_i}{1 + \zeta_1 + \sum_{i=2}^{m} \lambda_1 \lambda_2 \ldots \lambda_{i-1} \zeta_i} + 2 \frac{M[y]}{a^2} - 1 \right\} - M^2[y],$$

where, as before, λ_l and ζ_l are determined from (17) with account being taken of (I).

We transform the first term in the braces. Then we denote it by B(m). It is easily taken to the form

$$B(m) = \frac{2 \sum_{i=0}^{m} i^4 \frac{m!^2}{(m-i)!(m+i)!}}{2 \sum_{i=0}^{m} \frac{m!^2}{(m-i)!(m+i)!}}.$$

The denominator is the same as in the analogous expression for M[y], andequals $2^{2m} m!^2 : (2m)!$.

We now consider the series in the numerator.

$$\sum_{i=0}^{m} \frac{i^4}{(m-i)!(m+i)!} = m^4 A(m) -$$

$$-(2m^2 - 2m + 1) A(m-1) + A(m-2).$$

By substituting this expression in B(m) we get

$$B(m) = \frac{m(3m-1)}{4}.$$

It is obvious that all these formulas are valid for m > 1.

Finally, for m > 1 ($\eta / 4a^2 > 1$), we find that

$$D[y] = \frac{\eta}{16}\left(\frac{\eta}{2} - a^2\right).$$

For all $\eta / 4a^2 \leq 1$, we immediately obtain from (22) and (24) that

$$D[y] = \frac{1}{4} a^4$$

2. Mathematical Expectation and Dispersion of the Output Variable for a Uniform Distribution Law in the Case of a Piecewise-Linear Object Characteristic

For the case when the object characteristic is $y = |x|$, the general expression for the mathematical expectation of the output is found in [1]. We recall that

$$(\overline{\Delta y})_i = (i + \delta_{i,0}) a, \quad (i = 0, 1, \ldots), \quad \delta_{i,0} = \begin{cases} 1, & \text{if } i = 0, \\ 0, & \text{if } i \neq 0; \end{cases}$$

$$|(\overline{\Delta y})_i| = 2a; \; p_i = p = \int_{-\infty}^{-2a} \rho(v)\, dv;$$

$$r_1 = \frac{1}{2\left(q + \frac{1}{1-\lambda}\right)}; \quad r_0 = 2qr_1; \quad r_i = r_1 \lambda^{i-1},$$

where

$$\lambda = \frac{p}{q}; \quad M[y] = a \frac{q + \frac{1}{(1-\lambda)^2}}{q + \frac{1}{1-\lambda}}.$$

In the case of the uniform distribution of (23)

$$p = \begin{cases} \frac{1}{2} - \frac{a}{\eta} & \left(\frac{1}{2} > \frac{a}{\eta}\right), \\ 0 & \left(\frac{1}{2} \leqslant \frac{a}{\eta}\right); \end{cases} \quad q = \begin{cases} \frac{1}{2} + \frac{a}{\eta} & \left(\frac{1}{2} > \frac{a}{\eta}\right), \\ 1 & \left(\frac{1}{2} \leqslant \frac{a}{\eta}\right); \end{cases}$$

$$\frac{M[y]}{\eta} = \begin{cases} \frac{4\left(\frac{a}{\eta}\right)^2 + \frac{a}{\eta} + \frac{1}{2}}{4\frac{a}{\eta} + 2} & \text{for } \frac{a}{\eta} < \frac{1}{2}, \\ \frac{a}{\eta} & \text{for } \frac{a}{\eta} \geqslant \frac{1}{2}. \end{cases}$$

We compute the dispersion

$$-a^2 \left[\frac{q + \frac{1}{(1-\lambda)^2}}{q + \frac{1}{1-\lambda}} \right]^2 = \qquad -$$

$$D[y] = \sum_{i=-\infty}^{\infty} (\overline{\Delta y})_i r_i - M^2[y] = a^2 \frac{q + \sum_{i=1}^{\infty} i^2 \lambda^{i-1}}{q + \frac{1}{1-\lambda}} =$$

$$= a^2 \frac{q\lambda(1-\lambda^2) + \lambda}{(1-\lambda)^4 \left(q + \frac{1}{1-\lambda}\right)^2}.$$

By substituting p and q, we get

$$D[y] = \begin{cases} \frac{\eta^2}{4} \frac{\frac{1}{4} + \frac{a}{\eta} - 3\left(\frac{a}{\eta}\right)^2}{\left(1 + 2\frac{a}{\eta}\right)^2} & \text{for } \frac{a}{\eta} < \frac{1}{2}, \\ 0 & \text{for } \frac{a}{\eta} \geqslant \frac{1}{2}; \end{cases}$$

$$\frac{\sigma[y]}{\eta} = \begin{cases} \frac{1}{2} \frac{\sqrt{\frac{1}{4} + \frac{a}{\eta} - 3\left(\frac{a}{\eta}\right)^2}}{1 + 2\frac{a}{\eta}} & \text{for } \frac{a}{\eta} < \frac{1}{2}, \\ 0 & \text{for } \frac{a}{\eta} \geqslant \frac{1}{2}. \end{cases}$$

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THE ACTION OF RANDOM PROCESSES ON DISCONTINUOUS CONTROL SYSTEMS

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Questions of the statistical calculations of discontinuous (sampled-data) control systems are considered, these systems being acted upon by stationary random processes. We also consider one particular case when a nonstationary random process acts. It is shown that the method of statistical linearization can be used for the approximate calculation of nonlinear systems in the same form in which it is used for the calculation of continuous nonlinear systems.

INTRODUCTION

Today there is a heightened interest in sampled-data, or discontinuous, control systems. These include systems containing sampled-data elements of one form or another [1, 2], extrapolating devices [3, 4], and digital computers [5, 6].

Discontinuous control systems are superior to continuous ones in those cases when it is necessary to control a slowly varying process, since they require less energy for their operation.

Sampled-data systems can successfully be employed, and are employed, for the purpose of measurement, particularly telemetry; with this, they may easily be made multichannel devices.

Sampled-data systems can be used for the control of several objects by means of the same apparatus which a continuous system must use to control just one object.

The use of digital computers in control systems transforms them into discontinuous control systems since, by their nature, digital machines have to do with discrete information, with digits. Control systems with digital devices are very flexible since, by a simple change of program, they can be made capable of solving their problem under different conditions.

Discontinuous control systems of a quite broad class can be described from a single point of view. In this class fall those systems whose behavior is determined by the values of the input signals at discrete equispaced moments of time.

In [7] a method is given for describing the operation of sampled-data elements with arbitrary pulse forms. The essence of the method is that the actual sampled-data element is presented in the form of a series connection of ideal sampled-data elements which process pulses in the form of Dirac δ-functions (following one another at equally spaced moments of time with norms equal to the values of the element's input signal at the corresponding moments of time) and which form a link which transforms the δ-pulses to pulses of the necessary forms. The shaping link is linear, and is described by a certain weight function and transfer function.

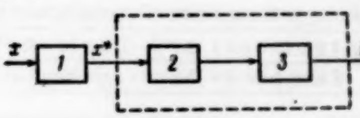

Fig. 1.

The series connection of the sampled-data element with a linear system forms a linear sampled-data circuit. The shaping link with the linear portion of the circuit forms the equivalent continuous portion (ECP) of the circuit (Fig. 1). The transfer function of the ECP is, obviously, the product of the transfer functions of the shaping link and the linear continuous portion of the circuit, and the weight function of the ECP is the convolution of the weight functions of the shaping link and the continuous portion.

If x(t) is the signal acting on the sampled-data circuit's input y (t) is the output signal and g(t) is the weight function of the ECP then, in the steady state of the circuit, the following relationship holds::

$$y(k\Delta t + \varepsilon\Delta t) = \sum_{n=-\infty}^{k} x(n\Delta t)\, g[(k-n)\Delta t + \varepsilon\Delta t]. \tag{1}$$

Here, the time t is given in the form $t = k\Delta t + \epsilon\Delta t$, where k is an integer, $\epsilon < 1$ and Δt is the interval of discreteness (quantization).

Application of the discrete Laplace transform [1] to (1) gives

$$Y^*(p, \varepsilon) = G^*(p, \varepsilon)\, X^*(p), \tag{2}$$

where Y^*, G^*, and X^* are the transforms of y, g, x, respectively. For example,

$$G^*(p, \varepsilon) = \sum_{m=0}^{\infty} g(m\Delta t + \varepsilon\Delta t)\, e^{-pm\Delta t}.$$

By analogy with continuous systems, $G^*(p, \epsilon)$ is called the transfer function of the sampled-data circuit. It may be shown that $G^*(p, \epsilon)$ is related to the transfer function (in the usual sense) of the ECP, G(p) by the relationship [8]

$$G^*(p, \varepsilon) = \frac{1}{\Delta t} \sum_{m=-\infty}^{\infty} G\left(p + i\frac{2\pi m}{\Delta t}\right) e^{\varepsilon \Delta t \left(p + i\frac{2\pi m}{\Delta t}\right)}. \quad (3)$$

If $\underline{p}$ is replaced by $i\omega$, one obtains the frequency characteristic of the sampled-data circuit, which may be interpreted as the ratio of the output signal to the input signal if the input signal is a function of the form $e^{i\omega n\Delta t}$.

Relationships (1) and (2) also hold for systems with feedback, except that, instead of $G^*(p, \epsilon)$ in (2), the system's transfer function enters and, instead of g(t) in (1), the properly defined weight function of system enters.

In the sequel, a time-domain function will always be denoted by a lower-case Latin letter, while its transform will be denoted by the corresponding upper-case letter; to distinguish discrete from continuous transforms, the discrete transforms will always be marked with a star.

1. Stationary Random Processes with Discrete Arguments and Their Actions on Linear Sampled-Data Systems

For the following exposition we shall need certain results from the theory of stationary (in the broad sense) random processes with discrete arguments. This theory is presented in great detail in the book of J. L. Doob [9]. In [10], the theory of discrete random processes is applied to a probabilistic computation of sampled-data systems. In contradistinction to [10], we shall here take as basic the spectral representation of stationary processes given in [9].

Each stationary random process $x(n\Delta t)$ with zero mathematical expectation, permits of a spectral representation

$$x(n\Delta t) = \frac{\Delta t}{2\pi} \int_{-\pi/\Delta t}^{\pi/\Delta t} e^{i\omega n\Delta t} du(\omega), \quad (4)$$

where the random process $u(\omega)$ has orthogonal increments, i.e.,

$$M[du(\omega_1)\overline{du(\omega_2)}] = \begin{cases} 0 & \text{for} \quad \omega_1 \neq \omega_2; \\ \frac{\Delta t}{2\pi} dF(\omega) & \text{for} \quad \omega_1 = \omega_2 = \omega. \end{cases} \quad (5)$$

Here, M is the symbol for mathematical expectation, and a superscript bar denotes the complex conjugate.

The correlation function of a discrete process can be represented in the form

$$R(n\Delta t) = \frac{\Delta t}{2\pi} \int_{-\pi/\Delta t}^{\pi/\Delta t} e^{i\omega n\Delta t} dF(\omega). \quad (6)$$

The function $F(\omega)$ [the same function as in (5)] is called the spectral function of the process, and its derivative with respect to ω, i.e., $S^*(\omega) = F'(\omega)$, is called the spectral density.

The spectral density can be presented in the form of a Fourier series whose coefficients are the values of the correlation function

$$S^*(\omega) = \sum_{n=-\infty}^{\infty} R(n\Delta t) e^{i\omega n\Delta t}. \quad (7)$$

It follows from (7) that the spectral density of a discrete process is a periodic function of ω with a period of $2\pi/\Delta t$.

A discrete random process may be obtained from a continuous one if one consider its values at discrete, equally spaced, moments of time.

If x(t) is a continuous stationary random process, then its values for $t = n\Delta t$ comprise a stationary discrete process. Let $S(\omega)$ be the spectral density of the continuous process. The spectral density $S^*(\omega)$ of the discrete process obtained from it can be computed by means of the relationship [11]

$$S^*(\omega) = \frac{1}{\Delta t} \sum_{m=-\infty}^{\infty} S\left(\omega + \frac{2\pi m}{\Delta t}\right), \quad (8)$$

which is easily obtained from (7).

If the stationary random process $x(n\Delta t)$, which can be presented in the form of (4), acts on the input of some linear sampled-data system with constant parameters, then the system's steady-state output, i.e., after damping of the transient response in the system, can be computed by taking account of (1) and (4). The output process will have the spectral representation

$$y(k\Delta t + \varepsilon\Delta t) = \frac{\Delta t}{2\pi} \int_{-\pi/\Delta t}^{\pi/\Delta t} e^{i\omega k\Delta t} G^*(i\omega, \varepsilon) du(\omega), \quad (9)$$

where $G^*(i\omega, \epsilon)$ is the system's frequency characteristic.

It follows from (9), when (5) is taken into account, that the output process' correlation function is

$$R_y(n\Delta t, \varepsilon\Delta t) = \frac{\Delta t}{2\pi} \int_{-\pi/\Delta t}^{\pi/\Delta t} e^{i\omega n\Delta t} S_y^*(\omega) d\omega, \quad (10)$$

where $S_y^*(\omega)$ is the spectral density of the process $\underline{y}$, related to the spectral density of process $x(n\Delta t)$ by the relationship

$$S_y^*(\omega, \varepsilon) = |G^*(i\omega, \varepsilon)|^2 S_x^*(\omega). \quad (11)$$

The dispersion of the process $\underline{y}$ can obviously be computed from the formula

$$\sigma_y^2(\varepsilon) = \frac{\Delta t}{2\pi} \int_{-\pi/\Delta t}^{\pi/\Delta t} |G^*(i\omega, \varepsilon)|^2 S_x^*(\omega) d\omega. \quad (12)$$

Formulas (10) and (12) were obtained from [10].

We turn nowto (8). It immediately follows from it th that the ideal sampled-data element transforms the high-frequency region of the spectrum to a low-frequency one. This leads to the consequence that the introduction of a sampled-data element into a continuous system increases the random errors. This fact is seen particularly clearly in the following example.

Let there be some continuous system, and let there act on it a stationary random process x(t) with spectral density $S(\omega)$. Let the system's frequency characteristic $K(i\omega)$ be such that for $|\omega| > \omega_0$, its modulus is so small that the dispersion of the output process can be computed by the formula

$$\sigma_C^2 \approx \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} S(\omega) \, |K(i\omega)|^2 \, d\omega. \tag{13}$$

Let us now connect this system to some sampled-data element whose pulse repetition period Δt satisfies the inequality

$$\Delta t < \frac{\pi}{\omega_0},$$

The dispersion of the output random process from the sampled-data circuit thus obtained is computed from (12). With the assumptions made relative to $K(i\omega)$ and Δt, the frequency characteristic of such a circuit will, according to (3), be

$$G^*(i\omega, \varepsilon) = \frac{1}{\Delta t} F(i\omega) K(i\omega) e^{i\omega\varepsilon\Delta t}, \tag{14}$$

where $F(i\omega)$ is the frequency characteristic of the sampled-data element's shaping link.

We assume that over the entire frequency range where $|K(i\omega)|$ still differs markedly from zero, $|F(i\omega)| / \Delta t$ differs little from unity. In fact, this assumption can be satisfied by choosing either a low-frequency part or a sufficiently small Δt, because $|F(i\omega)| / \Delta t \to 1$ for $\omega \to 0$ and differs greatly from unity only close to the frequencies $\omega = \pm\pi/\Delta t$.

Starting from this, we can write the following expression for the dispersion at the sampled-data circuit's output:

$$\sigma^2 \approx \frac{1}{2\pi} \int_{-\pi/\Delta t}^{\pi/\Delta t} |K(i\omega)|^2 \sum_{m=-\infty}^{\infty} S_x\left(\omega + \frac{2\pi m}{\Delta t}\right) d\omega. \tag{15}$$

By comparing (15) with (13), and taking into account that $\omega_0 < \pi/\Delta t$, we can infer that $\sigma_S^2 > \sigma_C^2$.

2. Action of a Nonstationary Signal on a Linear Sampled-Data System

Let a nonstationary signal of the following form act on a linear sampled-data system

$$\xi(t) = r(t) x(t), \tag{16}$$

where x(t) is a stationary random process and r(t) is a function of time.

The output process can be found by (1):

$$y(k\Delta t + \varepsilon\Delta t) = \sum_{n=-\infty}^{k} \xi(n\Delta t) g[(k-n)\Delta t + \varepsilon\Delta t]. \tag{17}$$

The process $\xi(n\Delta t)$ can obviously be presented in the form

$$\xi(n\Delta t) = r(n\Delta t) \frac{\Delta t}{2\pi} \int_{-\pi/\Delta t}^{\pi/\Delta t} e^{i\omega n\Delta t} du(\omega). \tag{18}$$

By substituting (18) in (17) we get

$$y(k\Delta t + \varepsilon\Delta t) = \frac{\Delta t}{2\pi} \int_{-\pi/\Delta t}^{\pi/\Delta t} du(\omega) e^{i\omega k\Delta t} \sum_{m=0}^{\infty} r[(k-m)\Delta t] g(m\Delta t + \varepsilon\Delta t) e^{-i\omega m\Delta t}. \tag{19}$$

Expression (19) is the spectral representation of the output process. It is a nonstationary process, since k enters into the integrand, and not only into the exponential factor.

We obtain the dispersion of the process y in the following form.

$$\sigma_y^2(k, \varepsilon) = \frac{\Delta t}{2\pi} \int_{-\pi/\Delta t}^{\pi/\Delta t} d\omega S_x^*(\omega) \left| \sum_{m=0}^{\infty} r[(k-m)\Delta t] g(m\Delta t + \varepsilon\Delta t) e^{-i\omega m\Delta t} \right|^2. \tag{20}$$

We consider two particular cases:

1. Let r(t) be a linear function of time:

$$r(t) = a + bt. \tag{21}$$

Then,

$$\sum_{m=0}^{\infty} [a + b(k - m)\Delta t]\, g[(m + \varepsilon)\Delta t]\, e^{-i\omega m\Delta t} =$$

$$= (a + bk\Delta t)\, G^*(i\omega, \varepsilon) + ib\frac{d}{d\omega}G^*(i\omega, \varepsilon).$$

If the system's frequency characteristic $G^*(i\omega, \varepsilon)$ is presented in the form

$$G^*(i\omega, \varepsilon) = P(\omega)\, e^{i\varphi(\omega)},$$

then, for the square of the modulus of the sum under the integral sign in (20), we obtain the expression

$$P^2[(a + bk\Delta t) + b\varphi']^2 + b^2P'^2,$$

where the "prime" denotes differentiation with respect to ω.

We obtain the following final expression for the dispersion:

$$\sigma_y^2(k, \varepsilon) = \frac{\Delta t}{2\pi}\int_{-\pi/\Delta t}^{\pi/\Delta t} d\omega S_x^*(\omega)\{P^2[(a + bk\Delta t) + b\varphi']^2 + b^2P'^2\}. \tag{22}$$

For computations by (22), it is necessary to differentiate the modulus and phase frequency characteristics with respect to ω. These operations may be implemented graphically.

2. Let

$$r(t) = e^{\alpha t}. \tag{21a}$$

In this case,

$$\sum_{m=0}^{\infty} e^{\alpha(k-m)\Delta t} g[(m + \varepsilon)\Delta t]\, e^{im\omega\Delta t} = e^{\alpha k\Delta t} G^*(\alpha + i\omega, \varepsilon)$$

and, consequently,

$$\sigma_y^2(k, \varepsilon) = e^{2\alpha k\Delta t}\frac{\Delta t}{2\pi}\int_{-\pi/\Delta t}^{\pi/\Delta t} d\omega S_x^*(\omega)\, |G^*(\alpha + i\omega, \varepsilon)|^2. \tag{23}$$

In order to compute the dispersion by this formula, it is necessary to reconstruct the system's frequency characteristic, since the presence of α would change all the time constants of the system's continuous part.

A formula analogous to (23) for the case of continuous systems was obtained by V. S. Krapivin [12].

Computations by the formulas obtained can be carried out in the case when the function f(t) is neither linear nor exponential. For this, it is necessary that r(t), in the neighborhood of the moment of time t_0 of interest, can be approximately presented in the form of a linear or exponential function. The size of this neighborhood can be estimated on the basis of the sum entering into (20). It must be such that the magnitude of the quantity under the summation sign outside of this neighborhood be so small that it would be possible to neglect it; at the same time, inside the neighborhood, r(t) must differ only slightly from a straight line or an exponential.

By using general relationship (20), one can compute the dispersion in cases when r(t) varies by a more general law than (21) or (21a).

3. Action of a Stationary Random Process on a Sampled-Data System which Contains a Nonlinear Element

We assume that the stationary random process of the form

$$x(n\Delta t + \varepsilon\Delta t) = \frac{\Delta t}{2\pi}\int_{-\pi/\Delta t}^{\pi/\Delta t} e^{i\omega n\Delta t} Q^*(\omega, \varepsilon)\, du(\omega) \tag{24}$$

acts on some continuous linear system with weight function k(t). The steady-state output process can be presented in the form

$$y(m\Delta t + \eta\Delta t) = \sum_{m=0}^{\infty}\Delta t\int_0^1 x[(m - n)\Delta t + \varepsilon\Delta t]\, k[n\Delta t + (\eta - \varepsilon)\Delta t]\, d\varepsilon. \tag{25}$$

By substituting (24) therein, we obtain the spectral representation of the output process

$$y(m\Delta t + \eta\Delta t) = \frac{\Delta t}{2\pi}\int_{+\pi/\Delta t}^{\pi/\Delta t} du(\omega)\, e^{im\omega\Delta t} P^*(\omega, \eta), \tag{26}$$

where

$$P^*(\omega, \eta) = \Delta\int_0^1 K^*(\omega, \eta - \varepsilon)\, Q^*(\omega, \varepsilon)\, d\varepsilon, \tag{27}$$

$$K^*(\omega, \eta - \varepsilon) = \sum_{m=0}^{\infty} k[m\Delta t + (\eta - \varepsilon)\Delta t]\, e^{-i\omega m\Delta t}. \tag{28}$$

We now consider the sampled-data circuit shown on Fig. 2. This circuit contains a nonlinear element. Such an element might be, for example, a relay element, an element with a limited linear zone, and other elements

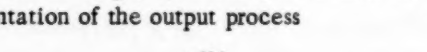

Fig. 2. Nonlinear sampled-data circuit. 1 is an ideal sampled-data element, 2 is a linear link, 3 is a nonlinear element, 4 is a linear link.

[13]. At the circuit's input acts a stationary random process $v(t) = m_v + v_0(t)$, where m_v is the mathematical expectation of the process $\underline{v}$ and $v_0(t)$ is a process with zero mathematical expectation. By using the results of section 1, we can easily find the spectral representation and, consequently, the correlation function $R_z(n\Delta t, \epsilon\Delta t)$ of the process $\underline{z}$ at the nonlinear element's input. It is also easy to compute the mathematical expectation m_z of the process $\underline{z}$. We shall assume that the process $v(t)$ is normally distributed. In this case, the correlation function $R_x(n\Delta t, \epsilon\Delta t)$ of the process $\underline{x}$ at the nonlinear element's output can be presented in the form of a series [14]

$$R_x(n\Delta t,\ \varepsilon\Delta t) = \sum_{l=1}^{\infty} \frac{a_l}{l!} \rho_z^l (n\Delta t,\ \varepsilon\Delta t). \tag{29}$$

Here, $\rho_z = R_z/\sigma_z^2$, σ_z^2 is the dispersion of process $\underline{z}$, and a_l are coefficients which depend on σ_z and m_z. In its turn, the mathematical expectation m_x of process $\underline{x}$ can be presented in the form

$$m_x = \chi(\sigma_z,\ m_z)\, m_z. \tag{30}$$

A simple and elegant method of calculating the coefficients a_l and χ, based on the use of the δ-function, is presented in [14].

For an element with a limited linear zone,

$$\chi = \frac{b}{a}\left\{1 + \frac{\sigma_z}{m_z}\left[\frac{a-m_z}{\sigma_z}\left(1 - \Phi\left(\frac{a-m_z}{\sigma_z}\right)\right) - \varphi\left(\frac{a-m_z}{\sigma_z}\right) - \right.\right.$$

$$\left.\left. - \frac{a+m_z}{\sigma_z}\left(1 - \Phi\left(\frac{a+m_z}{\sigma_z}\right)\right) + \varphi\left(\frac{a+m_z}{\sigma_z}\right)\right]\right\},$$

$$a_l = \sigma_z^2 \frac{b^2}{a^2}\left[\Phi^{(l-1)}\left(-\frac{a+m_z}{\sigma_z}\right) - \Phi^{(l-1)}\left(\frac{a-m_z}{\sigma_z}\right)\right]^2,$$

where $\underline{a}$ is the magnitude of the element's linearity zone, $\underline{b}$ is the limitation level and b/a is the slope of the linear part of the element's characteristic.

For a relay element,

$$\chi = \frac{1}{m_z} b\left[2\Phi\left(\frac{m_z}{\sigma_z}\right) - 1\right],$$

$$a_l = 4b^2\left[\Phi^{(l)}\left(-\frac{m_z}{\sigma_z}\right)\right]^2,$$

where $\underline{b}$ is the signal level at the relay element's output.

Here, everywhere $\quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt,$

and $\Phi^{(l)}(x)$ is the lth derivative of $\Phi(x)$ with respect to $\underline{x}$; $\varphi(x) = \Phi'(x)$.

The spectral density of process $\underline{x}$, starting from (29), can obviously be given in the form of a series

$$S_x^*(\omega,\ \varepsilon) = a_1 S_z^*(\omega,\ \varepsilon) + \sum_{l=2}^{\infty} \frac{a_l}{l!} \frac{1}{\sigma_z^{(2l+2)}} S_z^{*(l)}(\omega,\ \varepsilon), \tag{31}$$

where $S_z^*(\omega, \epsilon)$ is the spectral density of process $\underline{z}$, and $S_z^*(l)$ is the lth-order spectrum of the function $S_z^*(\omega, \epsilon)$.

Tables 1 and 2 give the values of the coefficients $a_l/l!$ for an element with a limited linearity zone and a relay element for $a = b = 1$, for various values of m_z and σ_z.

It is clear from the tables that in the overwhelming majority of the cases there holds the inequality $a_l/l! \ll a_1$, for $l = 2, 3, \ldots$ Thus, the terms with $l \geq 2$ in (31) can be ignored.

We turn back now to Fig. 2. Let link 2 have weight function $g(t)$ and link 4, weight function $k(t)$. Then the spectral density of process $\underline{z}$ will be

$$S_z^*(\omega,\ \varepsilon) = |G^*(i\omega,\ \varepsilon)|^2 S_v^*(\omega). \tag{32}$$

As a consequence of what has already been said about $S_x^*(i\omega, \epsilon)$, we have that

$$S_x^*(\omega,\ \varepsilon) \approx a_1 |G^*(i\omega,\ \varepsilon)|^2 S_v^*(\omega). \tag{33}$$

TABLE 1

m_z	σ_z	a_1	$a_2/2$	$a_3/6$	$a_4/24$
0	0.5	0.2220	0	0.00194	0
0	1.0	0.4640	0	0.03900	0
0	2.0	0.5860	0	0.08270	0
0.5	0.5	0.1760	0.0070	0.00272	0.00001
0.5	1.0	0.3900	0.0248	0.02280	0.00754
0.5	2.0	0.5540	0.0118	0.06950	0.00916
1.0	0.5	0.0625	0.0198	0	0.00167
1.0	1.0	0.2270	0.0594	0.00194	0.01310
1.0	2.0	0.4650	0.0494	0.03900	0.02650

TABLE 2

m_z	σ_z	a_1	$a_2/2$	$a_3/6$	$a_4/24$
0	0.5	0.6360	0	0.1060	0
0	1.0	0.6360	0	0.1060	0
0	2.0	0.6360	0	0.1060	0
0.5	0.5	0.2340	0.1170	0	0.03910
0.5	1.0	0.4950	0.0620	0.0495	0.03910
0.5	2.0	0.5970	0.0177	0.0865	0.01340
1.0	0.5	0.0116	0.0233	0.0175	0.00194
1.0	1.0	0.2340	0.1170	0	0.03910
1.0	2.0	0.4950	0.0620	0.0465	0.03910

From this we find that the approximate spectral representation of the process $\underline{x}$ has the form

$$x(n\Delta t + \varepsilon\Delta t) = \frac{\Delta t}{2\pi} \int\limits_{-\pi/\Delta t}^{\pi/\Delta t} e^{i\omega n\Delta t} \sqrt{a_1} G^*(i\omega, \varepsilon)\, du(\omega), \quad (34)$$

where $u(\omega)$ is the spectral process of $v_0(n\Delta t)$.

Process $\underline{x}$ acts on the input of linear link 4. We find the spectral representation of process $\underline{y}$ at the circuit's output in the form of (26).

By taking into account that, in the given case, $Q^*(i\omega, \varepsilon)$ is $\sqrt{a_1}\, G^*(i\omega, \varepsilon)$, we obtain

$$P^*(\omega, \eta) = \frac{1}{\Delta t} \sum_{n=-\infty}^{\infty} \sqrt{a_1} K\left(i\omega + i\frac{2\pi n}{\Delta t}\right) G\left(i\omega + i\frac{2\pi n}{\Delta t}\right) e^{i\eta\Delta t\left(\omega + \frac{2\pi n}{\Delta t}\right)}. \quad (35)$$

This expression is the frequency characteristic of the circuit in the case when the nonlinear element is replaced by an amplifier with gain equal to $\sqrt{a_1}$.

By using the representation of the nonlinear element in the form of a linear amplifier, we can easily calculate the probabilistic characteristics of the system with feedback. It is assumed for this that the process at the nonlinear element's input is normal. Such a method of calculating continuous systems is known as the method of statistical linearization, developed by I. E. Kazakov [13].

Thus, the statistical linearization method is also applicable to sampled-data systems. The computational methods for the discrete case are no different from those used in the continuous case, which are well-known. The coefficient $\sqrt{a_1}$ is computed for different types of nonlinear elements in [13].

CONCLUSIONS

In this paper we considered a number of questions dealing with the statistical computations of discontinuous control systems. We took as fundamental the spectral representation of stationary random processes with discrete arguments.

Starting from the relationship between the spectrum of a continuous process and that of the discrete process related to it, we found that under the same conditions the dispersion at the output will be larger in the sampled-data system than in the continuous system equivalent to it. This is a consequence of the fact that the sampled-data element transforms the high-frequency part of the input process spectrum into a low-frequency region.

The formulas obtained for the dispersion at the output of stationary systems permit the computations to be carried out in the cases when nonstationary processes act.

An investigation of the question of the action of a stationary process on a discrete system containing a nonlinear element led to the conclusion that the statistical linearization method is applicable to the discrete case, the computational methodology remaining the same as in the case of continuous systems.

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ERROR ACCUMULATION IN DIGITAL COMPUTATIONS

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The paper describes a method, based on diffusion theory, of estimating the errors of computing devices used in automatic control systems, as well as special-purpose and general-purpose digital computers. An investigation is made of the errors of cyclical computational processes, in which the successively accumulated elementary errors can be considered as independent random variables with an approximately normal distribution density.

Digital computing devices which solve recursive difference equations [1] are used in automatic control systems. In the operation of these devices, the computations are implemented cyclically, wherein the results of the previous cycle, or cycles, are the input data for the following cycle. For this, it is desirable to have the results obtained after each cycle. Recursive computational processes with output of intermediate data also arise frequently in the solution of various problems on general-purpose or special-purpose computers, for example, in computing tables of some function or other by recursion formulas.

With each computational cycle there ordinarily arise errors which gradually accumulate. This accumulation can occur either quite regularly or randomly. If the successively arising errors are virtually statistically independent, then they may be estimated by the use of probabilistic methods [2, 3]. The possibility of using these methods can be established [3] after an analysis of the statistical properties of the table of successive errors.

In many cases, the process of error accumulation amounts to the addition, to some total error which determines the quality of the entire solution and which exists prior to the given cycle, of an error resulting from the given cycle.

If the error attributable to the given cycle is the result of the effect of several independent sources of error (for example, rounding errors after several multiplications), then the resulting error of the cycle may have a probability density close to normal. Ordinarily, the mean value of the error in a cycle is zero. The dispersion σ^2 of the cycle's error is computed from an analysis of the table of successive errors of the cycles.

If one is given some limit a > 0 which must not be exceeded by the absolute magnitude of the total error Δ_n then, if the conditions stated above are met (addition of the successive errors of the cycles, statistical independence of these errors), the process of error accumulation can be considered as the random walk (diffusion) of a point between two absorbing barriers (Fig. 1). The ordinal numbers of the successive cycles are laid out along the

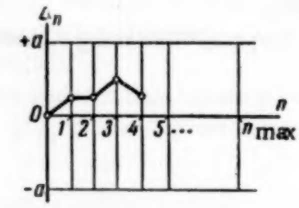

Fig. 1. Diffusion process with two absorbing barriers.

axis of abscissas on Fig. 1. The total error after the nth cycle is the sum of the individual cycle errors.

$$\Delta_n = \sum_{i=1}^{n} \delta_i.$$

The random walk begins at the point $\Delta_0 = 0$ (it is assumed that the initial data are given exactly). We shall assume that all the increments δ_i are independent random variables with identical normal distributions with dispersion σ^2 and zero mean value.

We now find the probability $P_0(n)$ that, in n cycles, the random walk will go beyond the barriers at least once.*

For a given n, the probability density of the random variable Δ_n has the form

$$f(x, n) = \frac{1}{\sqrt{n\sigma^2}\sqrt{2\pi}} \exp\left(-\frac{x^2}{2n\sigma^2}\right). \qquad (1)$$

Using the reflection method [4, 6], we can introduce a modified "distribution density"

$$f_1(x, n) = \sum_{s=-\infty}^{s=+\infty} (-1)^s \frac{1}{\sqrt{n\sigma^2}\sqrt{2\pi}} \exp\left\{-\frac{(x-2as)^2}{2n\sigma^2}\right\}, \qquad (2)$$

characterizing the probability distribution law of the complex event

$$f_1(x, n)\,dx = P\{x \leqslant \Delta_n \leqslant x + dx;$$
$$|\Delta_i| < a\} \quad (i = 1, 2, \ldots, n-1).$$

* For the case of continuous n, the problem, in a more general form, was solved by Bartlett [4, 5].

With this, consequently,

$$\int_{-a}^{a} f_1(x,\ n)\,dx = P\,\{|\,\Delta_n\,| < a;$$

$$|\,\Delta_i\,| < a\}\quad (i = 1,\ 2,\ \ldots,\ n-1).$$

$$\int_{-a}^{a} f_1(x,\ n)\,dx = P\,\{|\,\Delta_i\,| < a\}\qquad (i = 1,\ 2,\ \ldots,\ n).$$

The probability of interest to us, $P_0(n)$, equals

$$P_0(n) = 1 - \int_{-a}^{a} f_1(x,\ n)\,dx. \tag{3}$$

After some transformations, we can obtain from (2) and (3) that

$$P_0(n) = 4\sum_{s=0}^{\infty} F\left\{\frac{-a\,(2s+1)}{\sigma\sqrt{n}}\right\}(-1)^s =$$

$$= 4\left\{F\left(\frac{-a}{\sigma\sqrt{n}}\right) - F\left(\frac{-3a}{\sigma\sqrt{n}}\right) + F\left(\frac{-5a}{\sigma\sqrt{n}}\right) - \ldots\right\}, \tag{4}$$

where

$$F(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz.$$

$P_0(n)$, as a function of a/σ and $\underline{n}$, computed by (4), is shown on Fig. 2.

The method of estimating the error consists in finding such a value of $\underline{a}$ that, for a given number of cycles $\underline{n}$ and a dispersion σ^2 of the maximum possible error D_n (D_n is the maximum of all the Δ_i, $i = 1, 2, \ldots, n$), $P_0(n)$ will equal some definite, sufficiently small, quantity α (for example, $\alpha = 10^{-2}$, 10^{-3} or 10^{-4}). Thus, α is the confidence level for which

$$-D_n < \Delta_i < D_n \qquad (i = 1,\ 2,\ \ldots,\ n).$$

The dependence of D_n/σ on $\underline{n}$ for different α is shown in Fig. 3 on a semilogarithmic scale.

The graphs of Fig. 3 answer the question as to the practical limits of the errors Δ_i ($i = 1, 2, \ldots, n$) for a known dispersion σ^2 of the elementary errors (cycle errors). These graphs can be used for estimating the errors of any computing process in which there occurs accumulation (addition) of statistically independent successive errors with mean zero, dispersion σ^2, and a distribution close to a normal.

We now make an estimate of the error of computing the coordinates of successive equally spaced points on a

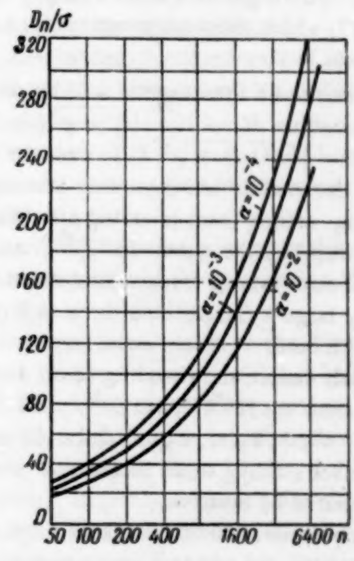

Fig. 3. Dependence of the maximum error D_n, divided by the mean-square deviation of the increment σ, on the number of cycles $\underline{n}$ and on the quantity α.

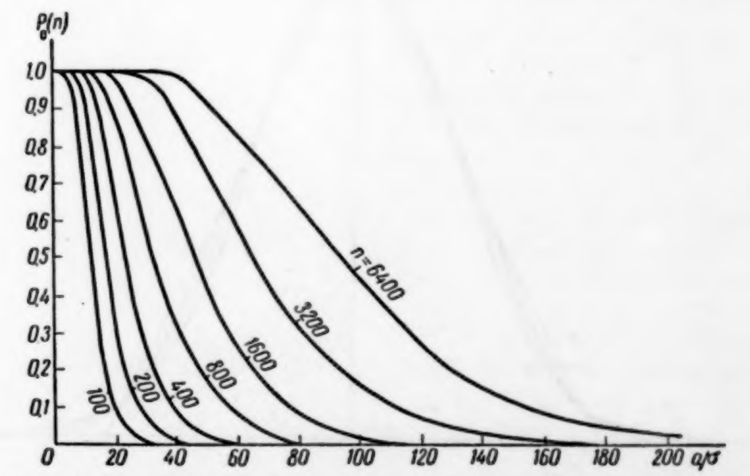

Fig. 2. The probability $P_0(n)$ of going beyond the barriers as a function of the number of cycles $\underline{n}$ and of the ratio of the distance to the barriers $\underline{a}$ to the mean-square deviation of the increment σ.

circle (with center at the origin) by means of the recursion formulas

$$x(n+1) = x(n)\cdot(1 - 2^{-2k-1}) \mp y(n)\cdot 2^{-k},$$

$$y(n+1) = y(n)\cdot(1 - 2^{-2k-1}) \pm x(n)\cdot 2^{-k},$$

which are obtained from the formulas

$$x(n+1) = x(n)\cos\omega \mp y(n)\sin\omega,$$

$$y(n+1) = y(n)\cos\omega \pm x(n)\sin\omega$$

after the substitutions

$$\cos\omega \approx 1 - 2^{-2k-1}, \quad \sin\omega \approx 2^{-k}, \qquad (5)$$

where $\underline{k}$ is a positive integer and ω is a constant angular increment. This algorithm is used in a special-purpose computer [7] which prepares programs for automatic milling machines.

The error to be investigated will be the deviation of the current radius $R_i = \sqrt{x^2(i) + y^2(i)}$ from the initial radius $R_0 = \sqrt{x^2(0) + y^2(0)}$ over the entire circumference of the circle. We shall only investigate the errors $\Delta R_i = R_i - R_0$ arising from rounding after multiplication (in binary form) by the quantities 2^{-2k-1} and 2^{-k}. The methodical error arising from the substitution in (5), for sufficiently large $\underline{k}$ (larger than three to five), is small and may be neglected.

We shall denote the rounding errors arising from the multiplications $x(n)2^{-2k-1}$, $y(n)2^{-k}$, $y(n)2^{-2k-1}$, and $x(n)2^{-k}$ by $\xi_1(n)$, $\xi_2(n)$, $\xi_3(n)$, and $\xi_4(n)$, respectively, where an error leading to an increase of the radius will be considered to be positive.

It can be shown that the error of a cycle equals

$$\delta_i = R_i - R_{i-1} \approx [\xi_1(i-1) + \xi_2(i-1)]\cos\beta + [\xi_3(i-1) + \xi_4(i-1)]\sin\beta,$$

where

$$\beta = \arctan\frac{y(i)}{x(i)}.$$

We shall assume that the computer operates in fixed point, that $|x| < 1$, $|y| < 1$, and that the machine uses an N-bit word after the binary point. A statistical analysis of the successive errors ξ, carried out for the given problem, showed that for $R_* > 2^{-N+4k+2}$, i.e., for sufficiently large N, all the quantities $\xi_1(n)$, $\xi_2(n)$, $\xi_3(n)$, $\xi_4(n)$ are statistically independent, and successive values do not depend on preceding ones. It turned out, with this, that they have a uniform distribution density p(z) (Fig. 4). The dispersion of such a distribution equals $2^{-2N}/12$, and the dispersion of the random variables $\delta_i = R_i - R_{i-1}$, the sum of the four independent random variables, equals

$$\sigma^2 = 2\frac{2^{-2N}}{12}\cos^2\beta + 2\,\frac{2^{-2N}}{12}\sin^2\beta = \frac{2^{-2N}}{6}.$$

The graphs of the distribution density p(z) of the quantity δ_i for various β, computed by a convolution integral, are shown on Fig. 5. Also constructed on the figure is the density curve for a normal distribution with the same dispersion. This curve differs slightly from the others, so that one can consider that all the δ_i are normally distributed with dispersion $2^{-2N}/6$.

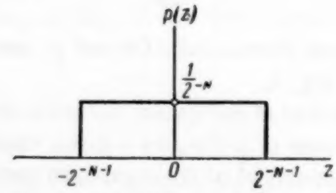

Fig. 4. Distribution density of the quantity ξ (n).

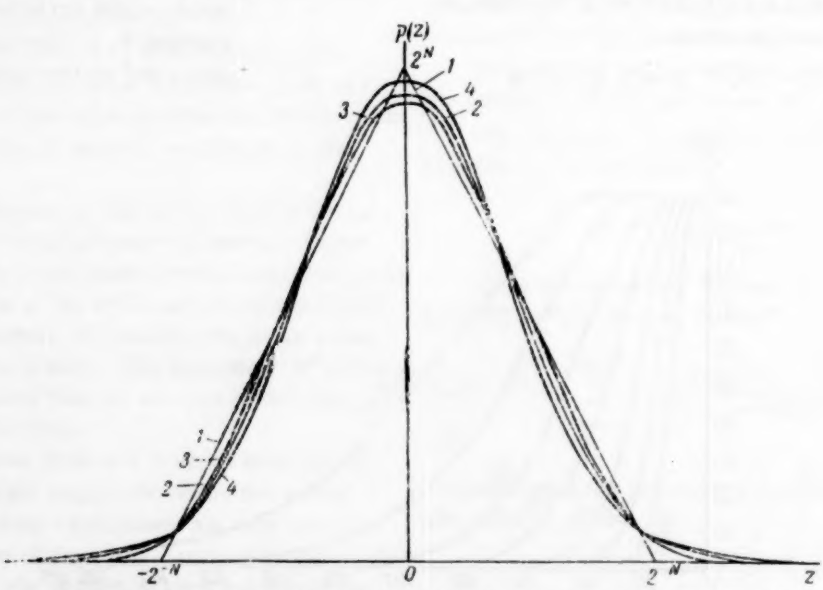

Fig. 5. Distribution density of the increment δ_i for various β: 1 is for $\beta = m\pi/2$; 2 is for $\beta = m\pi/2 \pm \pi/4$; 3 is for $\beta = m\pi/2 \pm \pi/8$, m = 0, 1, 2, 3; 4 is a normal distribution.

k	3	4	5	6	7	8	9	10
n	50	100	200	400	800	1600	3200	6400

The number of cycles n for the complete circle, equal approximately to $\pi 2^{k+1}$, changes by a factor of two when k changes by unity (cf., the table).

By using the graphs given on Fig. 3, one can determine the dependence of D_n on k. For example, for $k = 7$ and for a selected $\alpha = 10^{-3}$ the maximum error

$$|\Delta R_i|_{\max} = \left|\sum_{j=1}^{i} \delta_j\right|_{\max} = D_n \quad (i = 1, 2, \ldots, n)$$

equals approximately $(40)2^{-N}$. It is of interest to compare this value with the (theoretically) maximum possible error ΔR_n. Since the maximum value of $R_i - R_{i-1}$, for different β, varies from 2^{-N} to $\sqrt{2}\cdot 2^{-N}$ and, on the average, equals $(1.3)2^{-N}$, then the theoretically possible error ΔR_n equals $1.3n2^{-N} = (1040)2^{-N}$, which is 26 times the value obtained for D_n.

Instead of using graphs to determine D_n, one can use the well-known tables of the function $y = F(u)$. By substituting α instead of $P_0(n)$ and D_n instead of a in (4), and by discarding all except the first term, we get the equation

$$\alpha = 4F\left(\frac{D_n}{\sigma\sqrt{n}}\right).$$

Knowing α, σ, and n, we can easily determine D_n from the tables.

The simplification of (4) just given is admissible for $D_n/\sigma\sqrt{n}$, greater than 1 to 1.5 since, with this, the sum of the discarded terms, as a fraction of the first term, does not exceed 0.01 to 0.00005. For small $P_0(n)$ (or α), this condition is always met.

In about the same way, one can estimate the accumulated errors in those cases when the normal distribution of the independent variables δ_i has a nonzero mean.

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EFFECT OF RATE FEEDBACK ON LOADED RELAY SERVOMECHANISM DYNAMICS

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The effect of velocity feedback on the free oscillations of a loaded relay servomechanism is considered. The servomechanism's linear portion is described by a complete second-order equation. The investigation is carried out by the point method, using the theory of point transformations [1].

The dependence of the amplitude and time of the autooscillations' half periods on the system's parameters, including the velocity feedback coefficient, is determined.

1. Introduction and Posing of the Problem

The investigation of the dynamics of relay servomechanisms whose linear parts are described by complete second-order equations, has been carried out until now for the cases when the servomechanism's controlling element has a symmetric nonlinear characteristic. In [2-5] there were considered servomechanisms with Z-shaped or looped characteristics and, in [6], characteristics with loops and insensitive zones.

Few works have been published [7] in which consideration is given to unsymmetric looped characteristics of loaded-servomechanism controlling elements.

Some automatic control systems use electrical servomechanisms in which, by means of special relay circuits, there is implemented a charge in direction of the velocity by the switching of the supply voltage of an independently-excited motor from 24 to 48 v. To decrease the static error due to static friction, the servomechanisms are implemented in the form of autooscillatory systems. The autooscillations arise due to the unsymmetric characteristic of the controlling relay, which has one normally closed contact. In the servomechanism whose schematic is shown in Fig. 1, both direct and velocity (position and rate) feedback are employed.

2. Equations of Motion

As is clear from Fig. 1, the total voltage applied to the circuit is

$$\Delta U = U_{fb} - U_{fb} - U_{vf}, \qquad (1)$$

where U_{in} is the input signal voltage, U_{fb} is the (direct) feedback voltage and U_{vf} is the velocity feedback voltage (from the tachometer generator 4).

Controlling relay R_1, connected in the tube's anode circuit, has two working contacts $1R_1$ and $2R_1$. With no

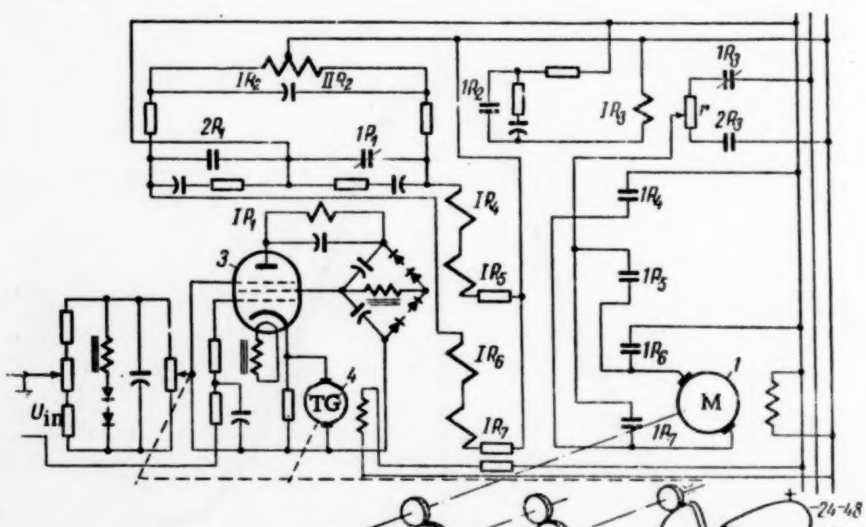

Fig. 1.

current in relay winding $1R_1$ (with negative grid bias), the relay closes contact 1R by means of a spring. With a positive grid bias, anode current will flow through the tube, causing relay R_1 to operate and closing contact $2R_1$. In this case, when contact $1R_1$ closes, current relays R_4 and R_5 and distributor relay R_2 are simultaneously in action. If contact $2R_1$ closes, then current relays R_6 and R_7 and distributor relay R_2 will be simultaneously in action. It is necessary to mention that, with zero signal on the input, relay R_1 operates with a frequency of 20 cps. With this, relay R_2 closes its working contact $1R_2$ with the doubled frequency 40 cps, since it is switched-in twice as frequently as the current relays. When relay R_2 operates, its armature touches contact $1R_2$ and switches in coil $1R_3$ of current relay R_3. Operation of current relays R_4 and R_5 leads to a rotation of motor 1 in one direction, while operation of relays R_6 and R_7 cause it to turn in the other direction. By its operation, relay R_3 switches the motor armature from the 24 v source to the 48 v source. When the winding of relay $1R_3$ is disconnected, its contact $1R_3$ closes the motor armature circuit on that side of resistor r which is connected to part of the 24 v storage battery.

When relay R_3 operates, its contact $2R_3$ closes the motor armature circuit to the other part of resistor r, which is connected to the full voltage of the 48 v storage battery. With no input signal, the motor will still be rotated by virtue of the absence of a neutral position for relay R_1. With zero input signal, the process of switching relay R_2 occurs so rapidly that the contact portion of relay R_3, due to its own inertia, cannot operate, and relay R_3 remains unconnected. With this, current relays R_4, R_5 or R_6,R_7 operate, feeding 24 v to the motor armature. In this case, steering organ 2, set into motion by the servomechanism, will oscillate with a frequency of 20 cps and an amplitude of about 0.5°. When a command is applied, the contact of relay $2R_1$ is closed. In this case, the duration of the closure is sufficient to allow relay R_2, and then relay R_3, to operate. In operating, relay R_3 applies 48 v to the motor armature. With this, the motor will execute the command with maximal speed and torque. Thanks to the presence in the circuit of the tachometer generator feedback, the operation of relay R_1 will be accelerated. With this, the motor will be switched to 24 volts upon reaching the position given by the input signal. This will preclude the arising of a lengthy oscillatory process.

We now consider the servomechanism's free oscillations, i.e., its motion for $U_{in} = 0$. The error equation of (1) can be given in the form

$$\Delta U = -K_\alpha \alpha - K_\Omega \Omega_1, \tag{2}$$

where α is the angle of rotation of the reducer's output axis, Ω_1 is the motor's angular rotational velocity, K_α and K_Ω are proportionality factors.

The speed of the reducer's output shaft is related to the motor velocity by the relationship $\Omega = \Omega_1/K_r = d\alpha/dt$, where K_r is the reduction coefficient.

The equation of the electronic amplifier can be given in the form

$$U_a = K_a \Delta U, \tag{3}$$

where U_a is the voltage at the amplifier's output and K_a is the gain.

The equation of the relay block has the form

$$U = \Psi[U_a] = |U| F(U_a), \tag{4}$$

where U is the voltage applied to the motor circuit.

We take the motor's voltage supply to be constant and equal to 48 v, i.e., we do not take into account the action of the scheme for switching the supply voltage. Such an assumption significantly simplifies the solution of the problem without leading to any essential errors in the investigation, since it is based on the fact that switching of the supply voltage to 24 v occurs close to zero error, and turns out to be essential only for the autooscillatory mode. Consequently, in calculating the autooscillation parameters, it is necessary to consider the motor's supply voltage as equal to 24 v, while for the investigation of the system's transient motions one should take the supply voltage as equally 48 v.

The nonlinear function $F(U_a)$ is shown in Fig. 2, where $\left(\varepsilon + \frac{\delta}{2}\right)$ and $\left(\varepsilon - \frac{\delta}{2}\right)$ are the operate and release voltages, respectively, for the relays.

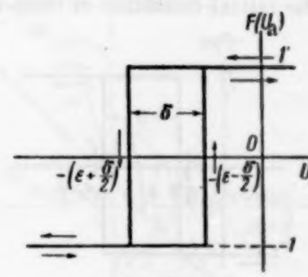

Fig. 2.

The equation for the motor's electrical circuit, if one neglects the armature's inductance, looks as follows:

$$U = IR + C_e \Omega_1, \tag{5}$$

where I is the armature current, R is the active impedance of the armature winding, $E_l = C_e \Omega_1$ is the motor's counter-emf and C_e is the motor constant.

If we take the load torque M_L to be proportional to the angle of rotation of the reducer's output axis α, i.e., $M_L = K_L \alpha$, then the equation of motor motion can be given in the form

$$J_{re} \frac{d\Omega}{dt} = C_M I K_r - K_L \alpha, \tag{6}$$

where C_M is the motor constant and J_{re} is the moment of inertia, referred to the reducer's output axis. By eliminating from (2)-(6) all the variables except $U_1 = K_a K_\alpha \alpha$, we get

$$A \frac{d^2 U_1}{dt^2} + B \frac{dU_1}{dt} + CU_1 = -F\left(U_1 + D \frac{dU_1}{dt}\right), \tag{7}$$

where

$$A = \frac{J_{\text{re}} R}{|U| C_{\text{M}} K_{\text{r}} K_{\alpha} K_{\text{a}}}, \quad B = \frac{C_{e} K_{\text{r}}}{|U| K_{\alpha} K_{\text{a}}},$$

$$C = \frac{R K_{\text{L}}}{|U| C_{\text{M}} K_{\text{r}} K_{\alpha} K_{\text{a}}}, \quad D = \frac{K_{\Omega} K_{\text{r}}}{K_{\text{a}}}.$$

We introduce the following dimensionless parameters:

$$t^{*} = \sqrt{\frac{C}{A}}\, t, \; x = CU_{1}, \; h = \frac{B}{2\sqrt{AC}},$$

$$k = D\sqrt{\frac{C}{A}}, \; \mu_{\varepsilon} = C\varepsilon, \; \Delta = C\delta. \tag{8}$$

Equation (7), written in dimensionless quantities, has the form

$$\frac{d^{2}x}{dt^{*2}} + 2h\frac{dx}{dt^{*}} + x = -\Phi\left(x + k\frac{dx}{dt^{*}}\right). \tag{9}$$

The function $\Phi(\sigma)$, where $\sigma = x + k dx/dt^{*}$, is shown on Fig. 3. This function is described by the equation

$$\Phi(\sigma) = \begin{cases} 1 \text{ for } \sigma \geqslant \mu_{\varepsilon} - \frac{\Delta}{2}, \text{ if } \Phi(\sigma_{0}) = 1, \\ -1 \text{ for } \sigma \leqslant \mu_{\varepsilon} + \frac{\Delta}{2}, \text{ if } \Phi(\sigma_{0}) = -1, \end{cases} \tag{10}$$

where σ_{0} is the initial condition at time $t = +0$.

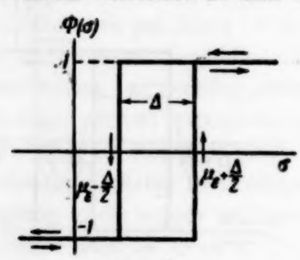

Fig. 3.

Figure 4 shows the block schematic of the servomechanism scheme described by (9). Thus, the system is characterized by four essential parameters $\underline{h}$, $\underline{k}$, μ_{ε}, and Δ.

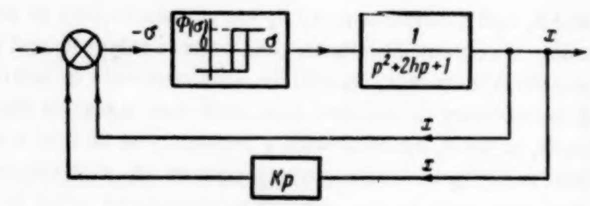

Fig. 4.

3. The Phase Plane

By setting $dx/dt^{*} = \dot{x} = y$, we write the equation of motion (9) in the form

$$\dot{x} = y, \quad \dot{y} + 2hy + x = -\Phi(x + ky). \tag{11}$$

We now consider the structure of the servomechanism's phase space. It follows, from the system of equations in (11), that the phase space of the system being investigated is a two-sheeted phase plane. On sheet I the system's motion is described by the differential equations

$$\dot{x} = y, \quad \dot{y} + 2hy + x = -1. \tag{12}$$

On sheet II,

$$\dot{x} = y, \quad \dot{y} + 2hy + x = 1. \tag{13}$$

From the system of equations (12) and (13), it follows that the system's equilibrium states are found at the points $x = -1$, $y = 0$ and $x = 1$, $y = 0$, which are stable foci (if $h < 1$) on stable nodes (if $h > 1$). Since, in actual systems, $\left|\mu_{\varepsilon} \pm \frac{\Delta}{2}\right| < 1$ and $\mu_{\varepsilon} > \frac{\Delta}{2}$, then the representative point, moving along a sheet of the phase plane, always reaches the switching line, never falling on a point

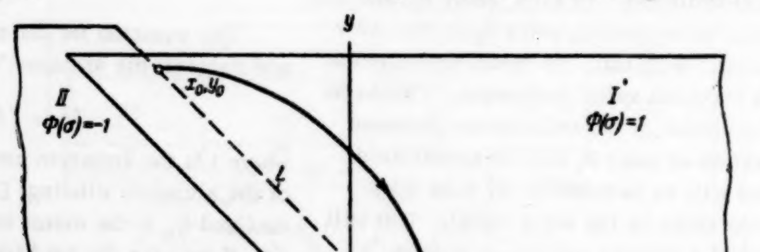

Fig. 5.

of rest. For the purpose of system stabilization, the damping coefficient $\underline{k}$ is ordinarily chosen to be positive. The motion of the representative point along the two-shetted phase plane occurs in the following manner. While moving along sheet $I\ [\Phi(x+ky)=1]$ (Fig. 5), the representative point, during some interval of time, falls on the line $L\left(\sigma=\mu_\varepsilon+\frac{\Delta}{2}\right)$ from the line $L_1\left(\sigma=\mu_\varepsilon-\frac{\Delta}{2}\right)$.

The system's phase space is not symmetric, since its equilibrium states are found at points on the axis $y = 0$ which are unsymmetrically placed with respect to the point $x = \mu_\varepsilon$, $y = 0$.

4. Point Transformation

The problem of seeking limiting cycles and of investigating the partitioning of the phase space leads to the study of the point transformation of line L into itself. This point transformation T is defined as $T = S^+S^-$. Transformation S^+ corresponds to a transition of the representative point, lying on line L, to a point lying on line L_1, and transformation S^- corresponds to a transition of the representative point lying on line L_1 to a point lying on line L. We shall find the analytic expressions for the transformations S^+ and S^-.

Sheet I of the Phase Plane $I\ [\Phi(\sigma)=-1]$

The case of small damping ($h < 1$). If, at time $t^* = 0$, the representative point begins its motion from point (x_0, y_0) of line L (cf., Fig. 5) and, during the time interval $t^* = t_1$, arrives at point (x_1, y_1) of line L_1, then the analytic expression for transformation S^+, in the form of a correspondence function $u_1 = u_1(v)$, in the case when $h < 1$, will have the form

$$u_1=\frac{\left(1+\mu_\varepsilon+\frac{\Delta}{2}\right)\cos w_1t_1-\left(1+\mu_\varepsilon-\frac{\Delta}{2}\right)e^{ht_1}}{\left(k-2h+\frac{1}{k}\right)\sin w_1t_1}w_1+$$

$$+\frac{\left(1+\mu_\varepsilon+\frac{\Delta}{2}\right)(h-k)}{k-2h+\frac{1}{k}}+\mu_\varepsilon+\frac{\Delta}{2}$$

$$v=\frac{-\left(1+\mu_\varepsilon-\frac{\Delta}{2}\right)\cos w_1t_1+\left(1+\mu_\varepsilon+\frac{\Delta}{2}\right)e^{-ht_1}}{\left(k-2h+\frac{1}{k}\right)\sin w_1t_1}w_1+ \tag{14}$$

$$+\frac{\left(1+\mu_\varepsilon-\frac{\Delta}{2}\right)(h-k)}{k-2h+\frac{1}{k}}+\mu_\varepsilon-\frac{\Delta}{2},$$

where $w_1^2 = 1-h^2$, $u_1 = x_0$, $v = x_1$, and t_1 is a parameter. It follows from (14) that

$$\frac{dv}{du_1}=\frac{f_1(t_1)}{f_2(t_1)},$$

where

$$f_1(t_1)=w_1\left(1+\mu_\varepsilon-\frac{\Delta}{2}\right)-\left(1+\mu_\varepsilon+\frac{\Delta}{2}\right)$$

$$(w_1\cos w_1t_1+h\sin w_1t_1)\,e^{-ht_1},$$

$$f_2(t_1)=-w_1\left(1+\mu_\varepsilon+\frac{\Delta}{2}\right)+\left(1+\mu_\varepsilon-\frac{\Delta}{2}\right)$$

$$(w_1\cos w_1t_1-h\sin w_1t_1)\,e^{ht_1},$$

$$\frac{d^2v}{du_1^2}=\frac{(k-2h+1/k)\,f_3(t_1)\sin^3 w_1t_1}{f_2^3(t_1)}, \tag{15}$$

where

$$f_3(t_1)=-\frac{2h}{w_1}\left[(1+\mu_\varepsilon)^2-\frac{\Delta^2}{4}\right]\sin w_1t_1+$$

$$+\left(1+\mu_\varepsilon-\frac{\Delta}{2}\right)^2e^{ht_1}-\left(1+\mu_\varepsilon+\frac{\Delta}{2}\right)^2e^{-ht_1}.$$

If the initial conditions are changed on sheet I, i.e., if x_0 and y_0 are changed, the representative point will be translated along sheet I during different time intervals.

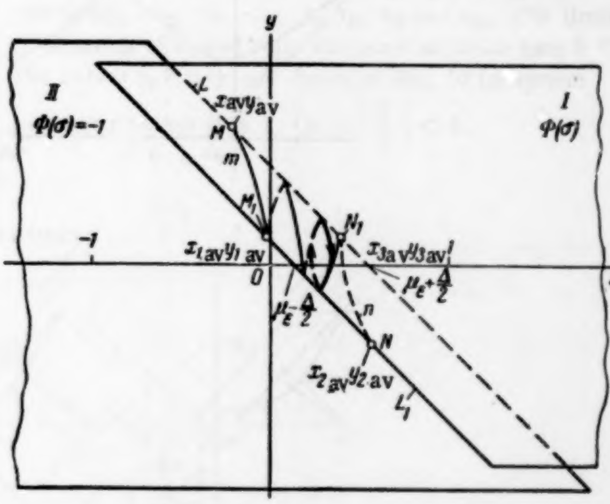

Fig. 6.

The transition time t_1 for the movement of the representative point from line L to line L_1 by a trajectory lying on sheet I and not going off on line L, i.e., the transition time along the "cylindrical" surface, lies in the interval $(0, t_{1s})$. The time T_{1s} is defined by the condition that trajectory $\underline{m}$ and line L be tangent at the point $M(x_{av}, y_{av})$ (cf., Fig. 6)

$$w_1\left(1+\mu_\varepsilon-\frac{\Delta}{2}\right)-\left(1+\mu_\varepsilon+\frac{\Delta}{2}\right)$$

$$(w_1\cos w_1t_{1s}+h\sin w_1t_{1s})\,e^{-ht_{1s}}=0. \tag{16}$$

The coordinates x_{av} and y_{av} of point M are also

defined by the condition that trajectory $\underline{m}$ and line L be tangent:

$$x_{av} = -k\frac{\left(1+\mu_\varepsilon+\frac{\Delta}{2}\right)}{k-2h+\frac{1}{k}} + \mu_\varepsilon + \frac{\Delta}{2}, \quad y_{av} = \frac{1+\mu_\varepsilon+\frac{\Delta}{2}}{k-2h+\frac{1}{k}}. \tag{17}$$

The transition time along a trajectory in sheet I and going off the L varies continuously from t_{1s} to π/w_1 as x_0 varies. An investigation of the function $v = v(u_1)$ for the case when $h < 1$ is given in Appendix I. The curve $v = v(u_1)$, given by the parametric equations (14) and (15), is defined for all t_1 from 0 to π/w_1. It is clear, from an analysis of the phase plane, that from the entire set of arms of the curve there should be retained only those arms which satisfy the following limits of coordinate variation:

$$u_1 \text{ from } \mu_\varepsilon + \frac{\Delta}{2} \text{ to } -\infty, \quad v \text{ from } x_{1\,av} \text{ to } \infty.$$

The curves $v = v(u_1)$ for values of system parameters such that $h < 1 \left|\mu_\varepsilon \pm \frac{\Delta}{2}\right| < 1$ are given in Fig. 7.

<u>The case of large damping ($h > 1$).</u> The transformation S^+ in the form of a correspondence function $u_1 = u_1(v)$, for the case when $h > 1$, has an analytic expression which may be obtained from (14) by replacing w_1, $\sin w_1t_1$ and $\cos w_1t_1$ by, respectively, $\underline{w}$, $\sinh wt_1$ and $\cosh wt_1$, where $w^2 = h^2 - 1$. In the case when $h > 1$, the transition time along a trajectory in sheet I which goes off line L varies within the interval (t_{1s}, ∞). The time t_{1s}, in the case when $h > 1$, has an expression which may be obtained from (16) by replacing w_1, $\sin w_1t_1$ and $\cos w_1t_1$ by, respectively, $\underline{w}$ $\sinh wt_1$ and $\cosh wt_1$. For the case when $h > 1$, the coordinates x_{av} and y_{av} are expressed, as in the case for $h < 1$, by (17). An investigation of the function $v = v(u_1)$ for the case when $h > 1$ is given in Appendix I. The limits of variation of u_1 and $\underline{v}$ are the same as in the case when $h < 1$. The curves of $v = v(u_1)$ for values of the parameters $h > 1, \left|\mu_\varepsilon \pm \frac{\Delta}{2}\right| < 1$ are given in Fig. 8.

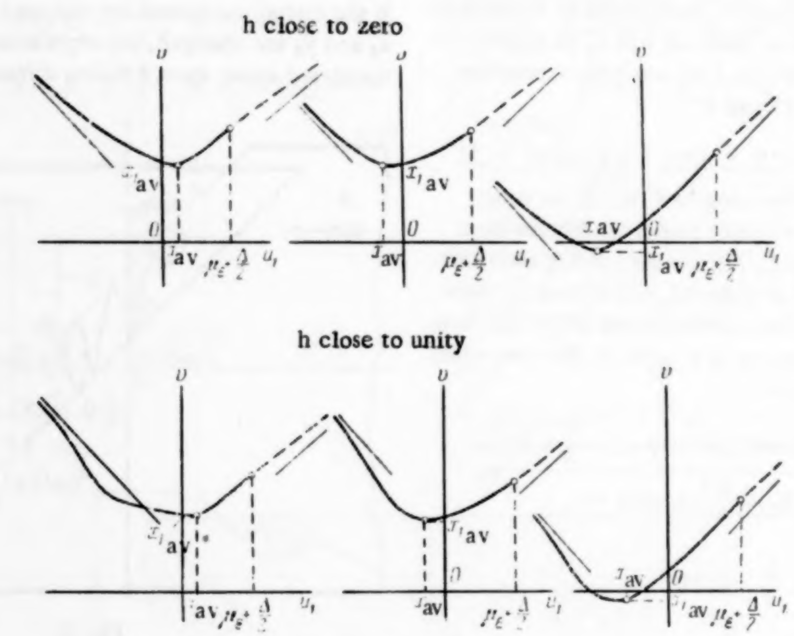

Fig. 7.

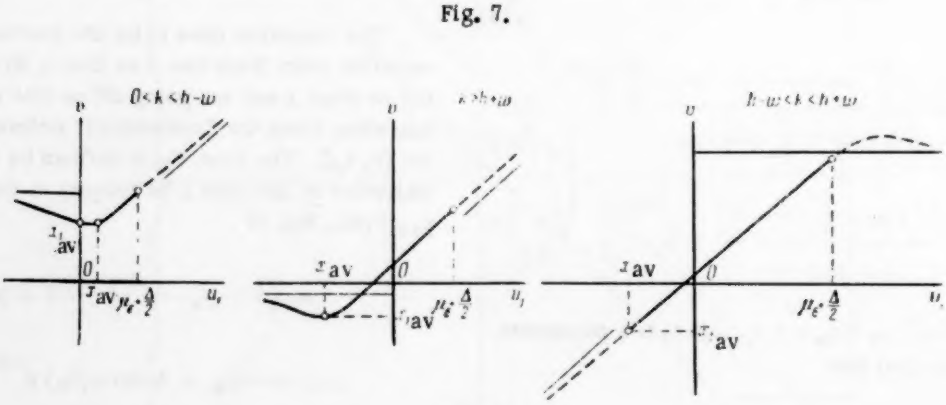

Fig. 8.

Sheet II of the Phase Plane $[\Phi(\sigma) = 1]$

The case of small damping ($h < 1$). If the representative point, at time $t^* = 0$, begins its motion from point (x_1, y_1) on line L_1 (Fig. 5) and, after a time interval $t^* = t_2$, arrives at the point (x_2, y_2) of line L, then the transformation S^- in the form of a correspondence function $u_2 = u_2(v)$ in the case $h < 1$ has an analytic expression which can be obtained from (14) by replacing v, u_1, t_1 and μ_ϵ by, respectively, $-u_2$, $-v$, t_2 and $-\mu_\epsilon$, where $u_2 = x_2$, $v = x_1$ and t_2 is a parameter.

The transition time t_2 for the movement of the representative point from line L_1 to line L along a trajectory in sheet II which does not leave line L_1 lies in the interval $(0, t_{2s})$. The transition time by a trajectory in sheet II which leaves line L_1 varies continuously from t_{2s} to π / w_1 as x_1 varies. The time t_{2s}, defined by the condition that trajectory $\underline{n}$ and line L_1 be tangent at the point $N(x_{2av}, y_{2av})$, has an expression, in the case when $h < 1$, which can be obtained from (16) by replacing μ_ϵ and t_{1s} by, respectively, $-\mu_\epsilon$ and t_{2s}. The coordinates x_{2av} and y_{2av} in the case $h < 1$ and in the case $h > 1$ have expressions which can be obtained from (17) by replacing x_{av}, y_{av}, and μ_ϵ by, respectively, $-x_{2av}$, $-y_{2av}$ and $-\mu_\epsilon$.

Investigation of the correspondence function $u_2 = u_2(v)$ in the case $h < 1$ can be carried out in analogy with that of the function $v = v(u_1)$ in the case when $h < 1$, which is given in Appendix, I, by replacing v, u_1, μ_ϵ, t_1, t_{1s} and t_{11} by, respectively, $-u_2$, $-v$, $-\mu_\epsilon$, t_2, t_{2s} and t_{22}.

From an analysis of the phase plane (Fig. 5), it is clear that of the entire set of arms of the curve $u_2 = u_2(v)$, there should remain only those arms which satisfy the following limits of coordinate variation:

$$u_2 \text{ from } x_{3av} \text{ to } -\infty, \; v \text{ from } x_{1av} \text{ to } \infty.$$

Curves of $u_2 = u_2(v)$ for system parameter values $h < 1, \left|\mu_\epsilon \pm \frac{\Delta}{2}\right| < 1$ are shown on Fig. 9. Also shown on Fig. 9 are curves $u_1 = u_1(v)$ for system parameter values $h < 1, \left|\mu_\epsilon \pm \frac{\Delta}{2}\right| < 1$.

The case of large damping ($h > 1$). The transformation S^- in the form of a correspondence function $u_2 = u_2(v)$ in the case $h > 1$ has an analytic expression which can be obtained from (14) by replacing v, u_1, μ_ϵ, w_1, t_1, $\sin w_1 t_1$ and $\cos w_1 t_1$ by, respectively, $-u_2$, $-v$, $-\mu_\epsilon$, w, t_2, $\sinh wt_2$ and $\cosh wt_2$. In the case $h > 1$, the time t_{2s} has an expression which can be obtained from (16) by replacing μ_ϵ, w_1, t_{1s}, $\sin s_1 t_{1s}$ and $\cos w_1 t_{1s}$ by, respectively, $-\mu_\epsilon$, w, t_{2s}, $\sinh wt_{2s}$ and $\cosh wt_{2s}$

Investigation of the function $u_2 = u_2(v)$ in the case $h > 1$ can be carried out in analogy to the investigation, given in Appendix II, of the function $v = v(u_1)$ in the case $h > 1$ by replacing v, u_1, μ_ϵ, t_1, t_{1s}, t_{13} and t_{14} by, respectively, $-u_2$, $-v$, $-\mu_\epsilon$, t_2, t_{2s}, t_{23} and t_{24}. The limits of variation of u_2 and $\underline{v}$ are the same as in the case $h < 1$. The curves $u_2 = u_2(v)$ are shown on Fig. 10 for system parameter values $h > 1, \left|\mu_\epsilon \pm \frac{\Delta}{2}\right| < 1$.

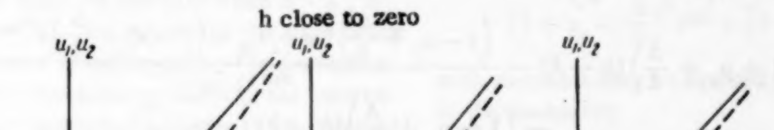

Fig. 9.

Figure 10 also shows curves of $u_1 = u_1(v)$ for system parameter values $h > 1$, $\left|\mu_\varepsilon \pm \frac{\Delta}{2}\right| < 1$.

System Autooscillations

It follows, from the diagrams of point transformation shown in Figs. 9 and 10 for the parameter values $h \neq 1$, $\left|\mu_\varepsilon \pm \frac{\Delta}{2}\right| < 1$, that for these system parameters, the curves of the correspondence function $u_1 = u_1(v)$ and $u_2 = u_2(v)$ must always have one point of intersection in the first quadrant. Consequently, there exists one stable limiting cycle, i.e., there are always autooscillations in the system.

These autooscillations can be established with the presence of a skipping mode (cf., Fig. 6). On the "cylindrical surface" between lines L and L_1, points MN_1NM_1 mark out the region in which a skipping mode occurs. From the conditions that the curves $u_1 = u_1(v)$ and $u_2 = u_2(v)$ intersect, in the case $h < 1$, we find the times t_{1a} and t_{2a} of the half periods of autooscillation in the case when $h < 1$. These conditions are written in the form

$$\frac{\left(1+\mu_\varepsilon+\frac{\Delta}{2}\right)e^{-ht_{1a}}-\left(1+\mu_\varepsilon-\frac{\Delta}{2}\right)\cos w_1 t_{1a}}{\sin w_1 t_a} w_1 + 2(h-k) =$$
$$= \frac{\left(1-\mu_\varepsilon-\frac{\Delta}{2}\right)e^{ht_{2a}}-\left(1-\mu_\varepsilon+\frac{\Delta}{2}\right)\cos w_1 t_{2a}}{\sin w_1 t_{2a}} w_1,$$
$$\frac{\left(1+\mu_\varepsilon+\frac{\Delta}{2}\right)\cos w_1 t_{1a}-\left(1+\mu_\varepsilon-\frac{\Delta}{2}\right)e^{ht_{1a}}}{\sin w_1 t_{1a}} w_1 + 2(h-k) = \tag{18}$$
$$= \frac{\left(1-\mu_\varepsilon-\frac{\Delta}{2}\right)\cos w_1 t_{2a}-\left(1-\mu_\varepsilon+\frac{\Delta}{2}\right)e^{-ht_{2a}}}{\sin w_1 t_{2a}} w_1$$

The amplitude of velocity autooscillation y is expressed, in the case when $h < 1$, by the equation $y_a = \frac{y_0 - y_1}{2}$ or

$$y_a = \frac{1}{2k\left(k-2h+\frac{1}{k}\right)}\Bigg[-\frac{\left(1+\mu_\varepsilon+\frac{\Delta}{2}\right)\cos w_1 t_{1a}-\left(1+\mu_\varepsilon-\frac{\Delta}{2}\right)e^{ht_{1a}}}{\sin w_1 t_{1a}} w_1 -$$
$$-\left(1+\mu_\varepsilon+\frac{\Delta}{2}\right)(h-k)-\frac{\left(1-\mu_\varepsilon+\frac{\Delta}{2}\right)\cos w_1 t_{2a}-\left(1-\mu_\varepsilon-\frac{\Delta}{2}\right)e^{ht_{2a}}}{\sin w_1 t_{2a}} w_1 -$$
$$-\left(1-\mu_\varepsilon+\frac{\Delta}{2}\right)(h-k)\Bigg], \tag{19}$$

where t_{1a} and t_{2a} are defined by (18). Analogously, for the case $h > 1$, the times of the autooscillation half periods t_{1a} and t_{2a} and the autooscillation amplitude are found from the equations

$$\frac{\left(1+\mu_\varepsilon+\frac{\Delta}{2}\right)e^{-ht_{1a}}-\left(1+\mu_\varepsilon-\frac{\Delta}{2}\right)\operatorname{ch} wt_{1a}}{\operatorname{sh} wt_{1a}} w + 2(h-k) =$$
$$= \frac{\left(1-\mu_\varepsilon-\frac{\Delta}{2}\right)e^{ht_{2a}}-\left(1-\mu_\varepsilon+\frac{\Delta}{2}\right)\operatorname{ch} wt_{2a}}{\operatorname{sh} wt_{2a}} w, \tag{20}$$

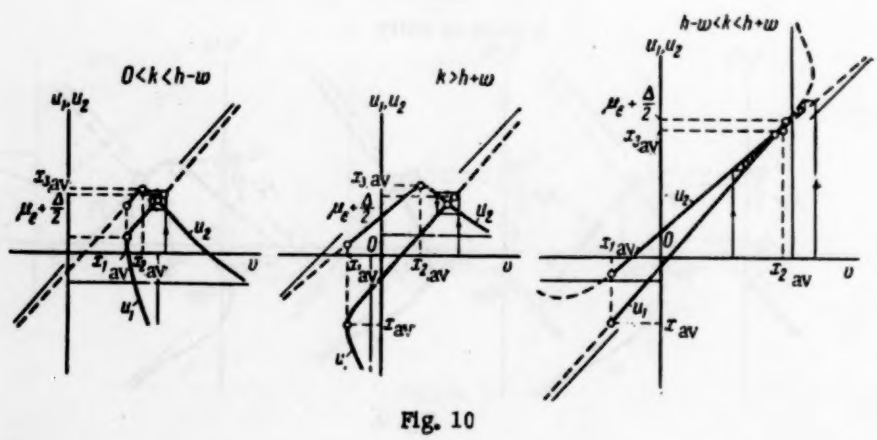

Fig. 10

$$\frac{\left(1+\mu_\epsilon+\frac{\Delta}{2}\right)\operatorname{ch} wt_{1a}-\left(1+\mu_\epsilon-\frac{\Delta}{2}\right)e^{ht_{1a}}}{\operatorname{sh} wt_{1a}}\,w+2(h-k)=$$

$$=\frac{\left(1-\mu_\epsilon-\frac{\Delta}{2}\right)\operatorname{ch} wt_{2a}-\left(1-\mu_\epsilon+\frac{\Delta}{2}\right)e^{-ht_{2a}}}{\operatorname{sh} wt_{2a}}\,w;$$

$$y_a=\frac{1}{2k\left(k-2h+\frac{1}{k}\right)}\Bigg[-\frac{\left(1+\mu_\epsilon+\frac{\Delta}{2}\right)\operatorname{ch} wt_{1a}-\left(1+\mu_\epsilon-\frac{\Delta}{2}\right)e^{ht_{1a}}}{\operatorname{sh} wt_{1a}}\,w-$$

$$-\left(1+\mu_\epsilon+\frac{\Delta}{2}\right)(h-k)-\frac{\left(1-\mu_\epsilon+\frac{\Delta}{2}\right)\operatorname{ch} wt_{2a}-\left(1-\mu_\epsilon-\frac{\Delta}{2}\right)e^{ht_{2a}}}{\operatorname{sh} wt_{2a}}\,w- \qquad (21)$$

$$-\left(1-\mu_\epsilon+\frac{\Delta}{2}\right)(h-k)\Bigg].$$

SUMMARY

With a looped relay characteristic of its controlling element, a loaded servomechanism whose linear portion is described by a complete second-order equation tends to an autooscillatory mode of operation for any parameter values satisfying the follow relationships:

$$h\neq 1,\quad \left|\mu_\epsilon\pm\frac{\Delta}{2}\right|<1,\quad k>0.$$

The author wishes to thank N. A. Fufaev and A. S. Alekseev for their valuable advice and comments in their reading and correcting of the manuscript.

Appendix I

Investigation of the Function $v = v(u_1)$ for the Case When $h < 1$

The function $v = v(u_1)$, for the case when $h < 1$, is defined by expression (14). We determine the signs of the derivatives dv/du_1 and d^2v/du_1^2 by (15). The derivative dv/du_1 is greater than zero when t_1 varies in the interval $(0, t_{1s})$ and is less than zero when t_1 varies in the interval $(t_{1s}, \pi/w_1)$ since

$$f_1(t_1)\begin{cases}<0 & \text{in interval } (0, t_{1S}),\\ >0 & \text{in interval } (t_{1S}, \pi/w_1).\end{cases}$$

$$f_2(t_1)<0 \quad \text{in interval } (0, \pi/w_1).$$

We now determine the sign of the second derivative. The sine of the angle w_1t_1 will be greater than zero when t_1 varies in the interval from 0 to π/w_1. The expression $k-2h+1/k$ is greater than zero, since $h < 1$. We investigate the sign of the function $f_3(t_1)$ [5]. The derivative $f_3'(t_1)$ equals

$$f_3'(t_1)=h\left\{-2\left[(1+\mu_\epsilon)^2-\frac{\Delta^2}{4}\right]\cos w_1t_1+\right.$$

$$\left.+\left(1+\mu_\epsilon-\frac{\Delta}{2}\right)^2e^{ht_1}+\left(1+\mu_\epsilon+\frac{\Delta}{2}\right)^2e^{-ht_1}\right\}.$$

Since, by the conditions of the problem $\left|\mu_\epsilon\pm\frac{\Delta}{2}\right|<1$ then, for $\cos w_1t_1 < 0$, the function $f_3'(t_1) > 0$, and for $\cos w_1t_1 < 0$, the expression $\left[(1+\mu_\epsilon)^2-\frac{\Delta^2}{4}\right]\cos w_1t_1>0$ and in the undetermined inequality

$$2\left[(1+\mu_\epsilon)^2-\frac{\Delta^2}{4}\right]\cos w_1t_1 \vee$$

$$\left(1+\mu_\epsilon-\frac{\Delta}{2}\right)^2e^{ht_1}+\left(1+\mu_\epsilon+\frac{\Delta}{2}\right)^2e^{-ht_1}$$

both members may be squared without changing the sign of the inequality

$$4\left[(1+\mu_\epsilon)^2-\frac{\Delta^2}{4}\right]^2\cos^2 w_1t_1 \vee \left(1+\mu_\epsilon-\frac{\Delta}{2}\right)^4e^{2ht_1}+$$

$$+2\left[(1+\mu_\epsilon)^2-\frac{\Delta^2}{4}\right]^2+\left(1+\mu_\epsilon+\frac{\Delta}{2}\right)^4e^{-2ht_1}$$

or

$$0<4\left[(1+\mu_\epsilon)^2-\frac{\Delta^2}{4}\right]^2\sin^2 w_1t_1+$$

$$+\left[\left(1+\mu_\epsilon-\frac{\Delta}{2}\right)^2e^{ht_1}+\left(1+\mu_\epsilon+\frac{\Delta}{2}\right)^2e^{-ht_1}\right]^2.$$

This is, the sign of indetermined inequality V is the sign of definite inequality <.

Thus, $f_3'(t_1)$ is greater than zero for all t_1. In the interval $(0, \pi/w_1)$ there exists, for h close to unity and $\left|\mu_\epsilon\pm\frac{\Delta}{2}\right|<1$, such a $t_1 = t_{11}$ that $f_3(t_{11}) = 0$ and

$$f_3(t_1)\begin{cases}<0 & \text{in interval } (0, t_{11}),\\ >0 & \text{in interval } (t_{11}, \pi/w_1).\end{cases}$$

for $\underline{h}$ close to zero and $\left|\mu_\epsilon\pm\frac{\Delta}{2}\right|<1$, the function

$f_3(t_1)$ is less than zero in the interval $(0, \pi / w_1)$. Consequently, the second derivative d^2v/du_1^2 can take the following signs:

$$\frac{d^2v}{du_1^2} = \begin{cases} > 0 \text{ at } (0, \pi/w_1) & \text{for } h, \sim 0 \quad \left|\mu_\varepsilon \pm \frac{\Delta}{2}\right| < 1, \\ \left.\begin{matrix} > 0 \text{ at } (0, t_{11}) \\ < 0 \text{ at } (t_{11}, \pi/w_1) \end{matrix}\right\} & \text{for } h, \sim 1 \quad \left|\mu_\varepsilon \pm \frac{\Delta}{2}\right| < 1. \end{cases}$$

For $t_1 \to 0$, the function $v \to \infty$ and $u_1 \to \infty$, but the derivative dv/du_1 tends to unity. For $t_1 \to 0$, the function $v = v(u_1)$ approximates to the asymptote

$$v = u_1 - \frac{\Delta}{k(k - 2h + 1/k)}.$$

For $t_1 \to \pi/w_1$, the functions $v \to \infty$ and $u_1 \to -\infty$, but the derivative tends to $-e^{-\pi h/w_1}$. The function $v = v(u_1)$, for $t_1 \to \pi/w_1$, tends to the asymptote

$$v = -e^{-\pi h/w_1} u_1 + \frac{(2h - k)(1 + e^{-\pi h/w_1})}{k - 2h + 1/k} +$$

$$+ \frac{\left(\mu_\varepsilon + \frac{\Delta}{2}\right) e^{-\pi h/w_1} + \mu_\varepsilon - \frac{\Delta}{4}}{k(k - 2h + 1/k)}.$$

Appendix II

Investigation of the Function $v = v(u_1)$ for the Case When $h > 1$

The function $v = v(u_1)$ and its derivatives dv/du_1 and d^2v/du_1^2 are defined by expressions which may be obtained from (14) and (15) by replacing w_1, $\sin w_1t_1$ and $\cos w_1t_1$ by w, $\sinh wt_1$ and $\cosh wt_1$, respectively.

We determine the sign of the derivative dv/du_1. The function $f_1(t_1)$ is less than zero for $0 < t_1 < t_{1s}$ and greater than zero for $t_{1s} < t_1 < \infty$, since $f_1(0) < 0$ and, for $t_1 = t_{1s}$, the function $f_1(t_{1s}) = 0$ and $f_1'(t_{1s}) > 0$. The function $f_2(t_1)$ is less than zero in the interval $0 < t_1 < \infty$, since $f_2(0)$ and $f_2'(t_1) = -(1 + \mu_\varepsilon - \Delta/2)e^{ht_1}\cosh wt_1 < 0$. Consequently, the derivative dv/du_1 is greater than zero in the interval $0 < t_1 < t_{1s}$ and less than zero in the interval $t_{1s} < t_1 < \infty$.

We determine the sign of the derivative d^2v/du_1^2. For $t_1 > 0$, the function $\sinh wt_1$ is greater than zero. The function $f_3(t_1)$, for t_1 equal to zero, is negative. For t_1 equal to t_{1s}, the function $f_3(t_1)$, equal to

$$\frac{(-\mathrm{ch}^2 wt_{1s} + 1)\left[(1 + \mu_\varepsilon)^2 - \frac{\Delta^2}{4}\right]}{w \,\mathrm{ch}\, wt_{1s} + h \,\mathrm{sh}\, w\, t_{1s}},$$

is also less than zero and, for $t_1 = \infty$, equals infinity. Moreover, the derivative $f_3'(t_1)$, for $t_1 = 0$, equal $hw\Delta^2$ and is positive. Further, the derivative $f_3'(t_1)$ equals zero for $t_1 = t_{13}$ and $t_1 = t_{14}$, whereby

$$\left(1 + \mu_\varepsilon + \frac{\Delta}{2}\right) e^{wt_{13}} = \left(1 + \mu_\varepsilon - \frac{\Delta}{2}\right) e^{ht_{13}}$$

and

$$\left(1 + \mu_\varepsilon + \frac{\Delta}{2}\right) e^{-wt_{14}} = \left(1 + \mu_\varepsilon - \frac{\Delta}{2}\right) e^{ht_{14}},$$

where $t_{14} < t_{13}$. Since the function $f_3(t_1)$ for $t_1 = t_{14}$ equals $\left[(1 + \mu_\varepsilon)^2 - \frac{\Delta^2}{4}\right](h + w) e^{-wt_{14}} \, [1 - e^{2wt_{14}}]$, and is less than zero, then $f_3(t_1)$ changes sign only once for some $t_{15} > t_{1s}$. In the case when $h > 1$, the expression $\left(k - 2h + \frac{1}{k}\right)$ is positive for $0 < k < h - \sqrt{h^2 - 1}$ or $k > h + \sqrt{h^2 - 1}$, and is negative for

$$h - \sqrt{h^2 - 1} < k < h + \sqrt{h^2 + 1}.$$

It thus follows from the analysis made that the derivative d^2v/du_1^2 may assume the following signs:

$$\begin{aligned} > 0 \text{ for } &\begin{cases} 0 < k < h - w & \text{or} \quad k > h + w \quad \text{for } \; 0 < t_1 < t_{13}, \quad t_{1s} < t_1 < t_{15}, \\ h - w < k < h + w & \qquad\qquad\qquad\quad\; \text{for } \; t_{15} < t_1 < \infty; \end{cases} \\ < 0 \text{ for } &\begin{cases} 0 < k < h - w & \text{or} \quad k > h + w \quad \text{for } \; t_{15} < t_1 < \infty, \\ h - w < k < h + w & \qquad\qquad\qquad\quad\; \text{for } \; 0 < t_1 < t_{1s} \quad t_{1s} < t_1 < t_{15}. \end{cases} \end{aligned}$$

For $t_1 \to 0$, the functions $v \to \infty$ and $u_1 \to \infty$, for $0 < k < h - w$ or $k > h + w$, and $v \to -\infty$ and $u_1 \to -\infty$ for $h - w < k < h + w$. The derivative dv/du_1 tends to unity for $t_1 \to 0$. For $t_1 \to 0$, the function $v = v(u_1)$ approximates to the asymptote $v = u_1 - \Delta/k(k - 2h + 1/k)$. For $t_1 \to \infty$, the function u_1 tends to ∞ for $h - w < k < h + w$ and to $-\infty$ for $0 < k < h - w$ or $k > h + w$, and the function

$$v \to \frac{-k - \left(\mu_\varepsilon - \frac{\Delta}{2}\right)(h + w) + (h - w) + \left(\mu_\varepsilon - \frac{\Delta}{2}\right)\frac{1}{k}}{k - 2h + 1/k}.$$

For $t_1 \to \infty$, the function $v = v(u_1)$ approximates to the asymptote

$$v = \frac{-k - \left(\mu_\varepsilon - \frac{\Delta}{2}\right)(h + w) + (h - w) + \left(\mu_\varepsilon - \frac{\Delta}{2}\right)\frac{1}{k}}{k - 2h + 1/k}.$$

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ON BINARY SIGNAL CORRECTOR SCHEMES

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Brief descriptions are given of the methods of constructing and correcting signal systems which were developed by M. A. Gavrilov [1, 2, 3], R. R. Varshamov [4, 5], and R. W. Hamming [6]. The structural formulas of the corrector scheme for each of the correction methods described are found. Corrector schemes are contrived.

One of the methods of providing for correct transmission of discrete signals entails the use of a corrector, which may be connected anywhere in the transmission channel.

We shall assume that the correctors are built of diodes and relay-contact elements.

Every discrete signal can be presented in the form of sequences of zeroes and ones. The number of binary digits (bits) in a signal is called its length. There are 2^n different signals of length n. In this set, one defines the concept of the distance between any pair of codes α and β as the number of noncoincident bits in them, i.e.,

$$\rho(\alpha, \beta) = \sum_{i=1}^{n} |\alpha_i - \beta_i|,$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$.

It is well known [6, 3] that, in a system capable of correcting d errors (i.e., d simultaneous inverted bits), the distance between any two codes in the system must be not less than 2d + 1.

Various methods of constructing such signal systems and implementing error correction are known [4-7]. Here, we shall consider only three of them.

Gavrilov's method of signal correction. Let a set N of working signals be so constructed that any two elements of N are separated by a distance not less than D = 2d + 1. There is established such a correspondence between the signals in N and the set M of all 2^n signals that each signal α in N corresponds to the same signal α in M and that all signals $\alpha' \in M$, whose distance from $\alpha \in M$ is not greater than d correspond to the same signal $\alpha \in N$. Then, instead of correcting the signal α', one can require that each α' be executed as the signal α which corresponds to it

This is a convenient method in that it can be realized for correcting any number of errors d that is desired.

Hamming's method for signal construction and correction. This method is applicable for the correction of single errors only. The construction of the set N is carried out according to definite rules, and signal correction (i.e., error detection) reduces to the verification that these rules are satisfied.

Let it be required to construct a set N of working signals which contains not more than k signals. There is then a number l which satisfies the inequality $k \leq 2$. Then, from the relationship

$$2^t - t - 1 \geqslant l \geqslant 2^{t-1} - (t-1) - 1$$

t is determined, and the signal length is found to be $n = l + t$. t bits of the signal (specifically, reading from left to right, the first, second fourth, eight, sixteenth, etc., i.e., the bits in positions $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, ..., 2^{t-1}) are called the check bits, while the remaining l bits are the information bits. The information bit positions are filled by all the possible combinations of zeroes and ones. The check bit positions are filled in accordance with the following rules.

In bit position $2^0 = 1$ there is placed a bit which will give a sum of zero modulo two to the bits in positions 2q + 1 (q = 0, 1, ...).

In bit position $2^1 = 2$ there is placed a bit which will give a sum of zero modulo two to the bits in positions 4q + 2 and 4q + 3 (q = 0, 1, ...).

In bit position $2^2 = 4$ there is placed a bit which will give a sum of zero modulo two to the bits in positions 8q + 4, 8q + 5, 8q + 6, and 8q + 7 (q= 0,1. ...).

In general, in bit position 2^i there is placed a bit which will give a sum of zero modulo two to the bits in positions $2^{i+1}q + 2^i$, $2^{i+1}q + 2^i + 1$, ..., $2^{i+1}q + 2^{i+1} - 1$ (q = 0,1, ...).

We now give an example of the construction of the set N by Hamming's method. Let k = 4, then $l = 2$, t = 3, and n = 5. The signals of set N are given in Table 1.

It is easily verified that the distance between any two signals is not less than 3.

Error-checking is executed by verifying the rules for the construction of set N. All three verifications are made; if the results of the verifications are denoted by the letters ξ, ζ and η, respectively, then the bit position in error in the signal is given by the binary number $\xi \zeta \eta$ (if one takes 1 = 100...0, 2 = 010...0, 4 = 00100...0). For example,

TABLE 1

Bit position	1	2	3	4	5
Position weight	2^0	2^1	2^2	2^3	2^4
α_1	0	0	0	0	0
α_2	1	0	0	1	1
α_3	1	1	1	0	0
α_4	0	1	1	1	1

if the signal sent were $\alpha_3 = (11100)$ and the signal received were $\alpha_3' = (11101)$, then $\xi = 1$, $\zeta = 0$ and $\eta = 1$. The bit position of the error is given by the number $\xi\zeta\eta$, equal, in this case, to 101 = 5.

Varshatov's method for signal construction and correction. The set N of working signals is so constructed that its elements form an additive group with respect to the operation of bit-by-bit addition modulo two, i.e., with respect to the operation defined by the relationship

$$a \dotplus b = (a_1 \dotplus b_1, a_2 \dotplus b_2, \ldots, a_n \dotplus b_n),$$

where $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_n)$ and the sign "$\dotplus$" is to be understood as addition modulo two.

Varshamov [4, 5] developed a methodology for constructing such sets N for any given code distance D. We limit ourselves here to just a brief description of this methodology.

Let it be required to construct a set N of signals containing not less than k elements. We determine the least integer l satisfying the relationship $k \leq 2^l$. We then find the least integer t satisfying the relationship

$$1 + C_{n-1}^1 + C_{n-1}^2 + \ldots + C_{n-1}^{D-2} < 2^t.$$

Then, $n = l + t$.

The first l bits in the signal are the information bits, the remaining t bits being the check bits.

The construction is executed as follows. One first constructs a unit matrix of l rows and l columns, to which one appends a second matrix of l rows and t columns. With this, the rows of this latter matrix must satisfy the following conditions: the number of ones in each row is not less than D-1; the number of ones in each sum (modulo two) of any two rows is not less than D-2 and, in general, the number of ones in each sum (modulo two) of any i rows is not less than D-i, where i assumes the values i = = 1, 2, . . . , D-1..

The set of l signals thus constructed is called a basis. The entire set of working signals is comprised of the set of basis signals plus all the possible sums (modulo two) of any 2,3,. . . , l basis signals (including the sum of any signal with itself, i.e., the null signal).

Example. Let k = 4 and D = 3, so that l = 2. We now determine t. For D = 3, we get

$$2^t - t - 1 \geqslant l$$

From this, t = 3 and, consequently, n = 5. We construct the unit matrix and append to it a matrix of two rows and three columns, in each row of which there is not less than two ones, the sum of these two rows containing not less than one "1"

10 011
01 111

The set N is made up of the two basis signals, their sum (modulo two) and the null signal. It is given in Table 2 (and coincides with the signals given in Table 1).

TABLE 2

α_1	00 000
α_2	10 011
α_3	11 100
α_4	01 111

R. R. Varshamov developed [5] a method for correcting d errors if the system of working signals has a code distance of D = 2d + 1. We limit ourselves here to a description of the method for correcting unitary errors, i.e., the case when D = 3.

Let there be received some signal α.

If follows, from the method of constructing the set N, that the information bits of all elements of N exhaust all the 2^l different sequences of binary numbers. Therefore, for each received signal there can be found, among the elements of set N, one element whose information bits coincide with the information bits of signal α. We denote this signal by β. We compare their check bits. It can happen that their check bits coincide exactly, i.e., $\alpha \equiv \beta$, and this means that the transmitted signal was received without distortion (error).

As was shown by the author of the method, if a unitary error changes one of the check bits of the transmitted signal, then the check bits of signal α will differ by one bit from the corresponding bits of signal β, where the bit position of the error is the bit position of the noncoinciding bit. If the unitary error changed one of the information bits of the transmitted signal, then the check bits of signal α will differ from the same bits of signal β by more than one bit, i.e., $\alpha \dotplus \beta = \gamma$.

With this, the information bits of γ will all be zero, and there will be more than one "one" among the check bits. To determine the bit position of the error, one seeks, among the signals of set N, a signal with the same check bits as γ (as was shown in [6], if the transmitted signal underwent no more than one distortion, then such a signal must of necessity be found). We denote it by δ. γ and δ differ by just one bit. The position of the latter gives the position of the error in signal α.

We give an example. Let the signal $\alpha_2 = (10011)$ be transmitted. We assume that the signal $\alpha = (10010)$ is received, i.e., that an error distorted the last check bit. From the table we find the unique signal α_2 with the same information bits as in α. Their check bits differ in one

place (bit position). The position of the differing bits is the fifth one; consequently, the error distorted the fifth bit, which was in fact the case.

We now assume that the same signal α_2 was transmitted, but that the error distorted its second bit, i.e., α = (11011). From Table 2 we find the unique signal σ_3 = (11100) with the same information bits as α. Their check bits differ in three positions. We then compute the sum $\alpha \dot{+} \beta = \gamma$ = (00111).

In Table 2 there is a signal δ whose check bits would be the sequence 111. Obviously, $\delta = \alpha_4$. By comparing the information bits of γ and δ, we convince ourselves that the second bits do not coincide, i.e., that the error distorted the second bit, which was in fact the case.

With some modifications, this method can be extended to correct any number of simultaneous errors.

Synthesis of the correctors. We now determine the structure of the corrector for each method of correction implementation.

If each bit of a signal is simulated by the contact of some relay, then the signal itself will be simulated by the constituent units.

The corrector schemes are relay devices at whose inputs are applied distorted or correct signals from the constructed system of working signals and at whose outputs appear only correct signals (the distorted signals will have been corrected by the corrector). It is very important in this connection that the number of (simultaneous) errors in the signals be matched by the code distances.

It is obvious that such devices must be nonsequential. With identical systems of working signals, identical sets of constituents are applied to the correctors' input, and the circuits, in which one and the same set of elements operate, can be made identical. Therefore, the correctors' schemes can also be made identical, despite the fact that the sequences of operations to be executed by them are different. Obviously, this assertion is valid for any n and d.

We illustrate its validity by the example wherein n = 5 and d = 1.

Corrector for the Gavrilov method. We write the states of relays X_1, X_2, X_3, X_4, and X_5, corresponding to the signals of Table 2:

$$\begin{aligned}
&\alpha_1 - \bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4\bar{x}_5,\\
&\alpha_2 - x_1\bar{x}_2\bar{x}_3x_4x_5,\\
&\alpha_3 - x_1x_2x_3\bar{x}_4\bar{x}_5,\\
&\alpha_4 - \bar{x}_1x_2x_3x_4x_5.
\end{aligned}$$

We write all the signals at unit distance from α_1, α_2, α_3, and α_4, denoting them, respectively, by

$$\alpha_1^{\mathrm{I}}, \alpha_1^{\mathrm{II}}, \alpha_1^{\mathrm{III}}, \alpha_1^{\mathrm{IV}}, \alpha_1^{\mathrm{V}}, \alpha_2^{\mathrm{I}}, \alpha_2^{\mathrm{II}}, \ldots, \alpha_4^{\mathrm{V}}:$$

$$\begin{array}{llll}
\alpha_1^{\mathrm{I}} - x_1\bar{x}_2\bar{x}_3\bar{x}_4\bar{x}_5; & \alpha_2^{\mathrm{I}} - \bar{x}_1\bar{x}_2\bar{x}_3x_4x_5; & \alpha_3^{\mathrm{I}} - \bar{x}_1x_2x_3\bar{x}_4\bar{x}_5; & \alpha_4^{\mathrm{I}} - \bar{x}_1x_2x_3x_4\bar{x}_5;\\
\alpha_1^{\mathrm{II}} - \bar{x}_1x_2\bar{x}_3\bar{x}_4\bar{x}_5; & \alpha_2^{\mathrm{II}} - x_1x_2\bar{x}_3x_4x_5; & \alpha_3^{\mathrm{II}} - x_1\bar{x}_2x_3\bar{x}_4\bar{x}_5; & \alpha_4^{\mathrm{II}} - \bar{x}_1x_2x_3\bar{x}_4x_5;\\
\alpha_1^{\mathrm{III}} - \bar{x}_1\bar{x}_2x_3\bar{x}_4\bar{x}_5; & \alpha_2^{\mathrm{III}} - x_1\bar{x}_2x_3x_4x_5; & \alpha_3^{\mathrm{III}} - x_1x_2\bar{x}_3\bar{x}_4\bar{x}_5; & \alpha_4^{\mathrm{III}} - \bar{x}_1x_2\bar{x}_3x_4x_5;\\
\alpha_1^{\mathrm{IV}} - \bar{x}_1\bar{x}_2\bar{x}_3x_4\bar{x}_5; & \alpha_2^{\mathrm{IV}} - x_1\bar{x}_2\bar{x}_3\bar{x}_4x_5; & \alpha_3^{\mathrm{IV}} - x_1x_2x_3x_4\bar{x}_5; & \alpha_4^{\mathrm{IV}} - \bar{x}_1\bar{x}_2x_3x_4x_5;\\
\alpha_1^{\mathrm{V}} - \bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4x_5; & \alpha_2^{\mathrm{V}} - x_1\bar{x}_2\bar{x}_3x_4\bar{x}_5; & \alpha_3^{\mathrm{V}} - x_1x_2x_3\bar{x}_4x_5; & \alpha_4^{\mathrm{V}} - x_1x_2x_3x_4x_5.
\end{array}$$

We shall denote the corrector's outputs by Y_i (the state of the i th output coincides with the undistorted state of the i th input).

Output Y_1 must be high (must carry voltage) if the second (α_2) or third (α_3) signal was transmitted or if any of the signals at unit distance from α_2 or α_3^{I} was applied to the corrector input.

We denote by c_2 the sum of the constituents corresponding to α_2 and α_2^{I}, α_2^{II}, α_2^{III}, α_2^{IV}, α_2^{V}, and, by c_3, the sum of the constituents corresponding to α_3, α_3^{I}, α_3^{II}, α_3^{III}, α_3^{IV}, and α_3^{V}.

$$\begin{aligned}
c_2 &= \bar{x}_1\bar{x}_2\bar{x}_3x_4x_5 + x_1\bar{x}_2x_3x_4x_5 + x_1x_2\bar{x}_3x_4x_5 +\\
&+ x_1\bar{x}_2\bar{x}_3\bar{x}_4x_5 + x_1\bar{x}_2\bar{x}_3x_4\bar{x}_5 + x_1\bar{x}_2\bar{x}_3x_4x_5, +\\
+ c_3 &= \bar{x}_1x_2x_3\bar{x}_4\bar{x}_5 + x_1\bar{x}_2x_3\bar{x}_4\bar{x}_5 + x_1x_2\bar{x}_3\bar{x}_4\bar{x}_5 +\\
&+ x_1x_2x_3x_4\bar{x}_5 + x_1x_2x_3\bar{x}_4x_5 + x_1x_2x_3\bar{x}_4\bar{x}_5, \qquad (1)
\end{aligned}$$

which gives us

$$F_{(Y_1)} = c_2 + c_3. \qquad (2)$$

By introducing the analogous notation for c_4

$$\begin{aligned}
c_4 = {} & \bar{x}_1x_2x_3x_4x_5 + x_1x_2x_3x_4x_5 +\\
& + \bar{x}_1\bar{x}_2x_3x_4x_5 + \bar{x}_1x_2\bar{x}_3x_4x_5 +\\
& + \bar{x}_1x_2x_3\bar{x}_4x_5 + \bar{x}_1x_2x_3x_4\bar{x}_5 \qquad (1')
\end{aligned}$$

and by reasoning the same as in the derivation of $F_{(Y_1)}$, we get

$$\begin{aligned}
F_{(Y_2)} &= c_3 + c_4,\\
F_{(Y_3)} &= c_3 + c_4,\\
F_{(Y_4)} &= c_2 + c_4, \qquad (2')\\
F_{(Y_5)} &= c_2 + c_4.
\end{aligned}$$

Corrector for the Hamming method. The corrector must execute three verifications. Let the result of each of these be recorded by the auxiliary relays Z_1, Z_2, and Z_3. Then

$$F_{(Z_1)}' = x_1 \oplus x_3 \oplus x_5 = x_1x_3x_5$$

$$+ \bar{x}_1\bar{x}_3x_5 + x_1\bar{x}_3\bar{x}_5 + \bar{x}_1x_3\bar{x}_5,$$

$$F_{(Z_2)} = x_2 \otimes x_3 = x_2\bar{x}_3 + \bar{x}_2 x_3,$$
$$F_{(Z_3)} = x_4 \oplus x_5 = x_4\bar{x}_5 + \bar{x}_4 x_5. \qquad (3)$$

The circuit comprised of $\bar{z}_1\bar{z}_2\bar{z}_3$ indicates an absence of errors, the circuit $z_1z_2z_3$ indicates that an error occurred in the second bit, $\bar{z}_1z_2z_3$ indicated that an error occurred in the third bit, etc. On the basis of such arguments, we can arrive at the functional schematic shown on Fig. 1. Such a scheme was developed in practice and was described in [8]* but, due to the presence of feedback in it (from relay Z_1 to relay X_j), its operation is unstable. To eliminate the instability of circuit operation, one can transform it in the following way: at the circuit's outputs we supply the states of the input elements and the corresponding corrections to them, obtained by means of the auxiliary relay. Then, the action functions for the input circuits are written as follows:

$$\begin{aligned}
F_{(Y_1)} &= x_1(\bar{z}_1\bar{z}_2\bar{z}_3 + \bar{z}_1z_2\bar{z}_3 + \bar{z}_1\bar{z}_2z_3 + z_1z_2\bar{z}_3 + z_1\bar{z}_2z_3) + \bar{x}_1z_1\bar{z}_2\bar{z}_3,\\
F_{(Y_2)} &= x_2(\bar{z}_1\bar{z}_2\bar{z}_3 + z_1\bar{z}_2\bar{z}_3 + \bar{z}_1\bar{z}_2z_3 + z_1z_2\bar{z}_3 + z_1\bar{z}_2z_3) + \bar{x}_2\bar{z}_1z_2\bar{z}_3,\\
F_{(Y_3)} &= x_3(\bar{z}_1\bar{z}_2\bar{z}_3 + z_1\bar{z}_2\bar{z}_3 + \bar{z}_1z_2\bar{z}_3 + \bar{z}_1\bar{z}_2z_3 + z_1\bar{z}_2z_3) + \bar{x}_3z_1\bar{z}_2\bar{z}_3,\\
F_{(Y_4)} &= x_4(\bar{z}_1\bar{z}_2\bar{z}_3 + z_1\bar{z}_2\bar{z}_3 + \bar{z}_1z_2\bar{z}_3 + z_1z_2\bar{z}_3 + z_1\bar{z}_2z_3) + \bar{x}_4\bar{z}_1\bar{z}_2z_3,\\
F_{(Y_5)} &= x_5(\bar{z}_1\bar{z}_2\bar{z}_3 + z_1\bar{z}_2\bar{z}_3 + \bar{z}_1z_2\bar{z}_3 + \bar{z}_1\bar{z}_2z_3 + z_1z_2\bar{z}_3) + \bar{x}_5z_1z_2z_3.
\end{aligned} \qquad (4)$$

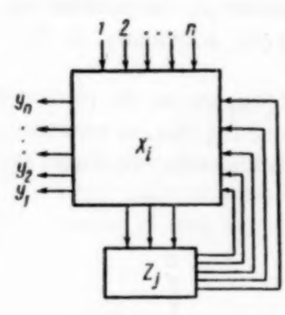

Fig. 1.

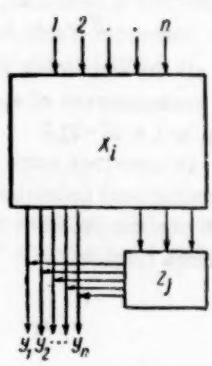

Fig. 2.

The functional schematic of such a corrector is shown on Fig. 2.

If the corrector is constructed in accordance with the structural formulas (3) and (4), then the corrector will contain three auxiliary relays, in addition to the five basic ones, and will be a two-cycle circuit.

Since we established that, in principle, the corrector is a nonsequential device, we carry out a number of transformations of structural formulas (4), specifically: we replace the letter z_i in them by the corresponding circuit paths of contacts x_j from (3). As a result of such substitutions, using the notation of (1), (1'), we obtain structural formulas (2), (2').

Corrector for the Varshamov method. The noncoinciding check bits in the received signal α and the signal from N with the same information bits as in α will be simulated by relays Z_1, Z_2, and Z_3. Relay Z_1, consequently, must be switched on if, when signal α is applied, one of the following four conditions holds: relays X_1 and X_2 do not operate and relay X_3 does operate; relays X_1 and X_3 do operate and relay X_2 does not; relays X_1 and X_2 do operate and relay X_3 does not; relays X_1 and X_3 do not operate and relay X_2 does, i.e.,

$$F_{(Z_1)} = \bar{x}_1\bar{x}_2x_3 + x_1\bar{x}_2x_3 + x_1x_2\bar{x}_3 + \bar{x}_1 x\,\bar{x}\,. \qquad (5)$$

By analogous reasoning, we can obtain the action functions for relays Z_2 and Z_3:

$$F_{(Z_2)} = \bar{x}_1\bar{x}_2x_4 + x_1\bar{x}_2\bar{x}_4 + x_1x_2x_4 + \bar{x}_1x_2\bar{x}_4,$$
$$F_{(Z_3)} = \bar{x}_1\bar{x}_2x_5 + x_1\bar{x}_2\bar{x}_5 + x_1x_2x_5 + \bar{x}_1x_2\bar{x}_5. \qquad (5')$$

If only some one relay Z is activated, then the error occurred in the check bits where, if relay Z_1 is activated, then the first check bit should be corrected, i.e., Y_3 must have voltage on it if relay X_3 is switched off, and conversely, and all the other outputs must be under voltage if the corresponding X relays are switched in. If only relay Z_2 is activated, then the output Y_4 is in the privileged position and , if the single relay Z_3 is activated, then this position will be held only by the output Y_5.

It is easily remarked that an error in the first information bit of any signal will lead to a noncorrespondence in the second and third check bits, i.e., relays Z_2 and Z_3 must be activated simultaneously. It is also easily verified that an error in the second information bit of any signal leads to a noncorrespondence of all three check bits, i.e., relays Z_1, Z_2, and Z_3 must be activated simultaneously. And, finally, if there are no errors in the signal, then there will be no noncorrespondences in the check bits, i.e., all the Z relays will be inactivated.

* In the circuit drawings certain annoying misprints occurred: in circuit (a) $\bar{x}_1$ should appear instead of x_1, and x_1 should appear instead of $\bar{x}_1$.

All that has been said so far can be written in the form of the following table:

If	$z_1\bar{z}_2\bar{z}_3$	then	$Y_1 = x_1$	$Y_2 = x_2$	$Y_3 = \bar{x}_3$	$Y_4 = x_4$	$Y_5 = x_5$
If	$\bar{z}_1z_2\bar{z}_3$	then	$Y_1 = x_1$	$Y_2 = x_2$	$Y_3 = x_3$	$Y_4 = \bar{x}_4$	$Y_5 = x_5$
If	$\bar{z}_1\bar{z}_2z_3$	then	$Y_1 = x_1$	$Y_2 = x_2$	$Y_3 = x_3$	$Y_4 = x_4$	$Y_5 = \bar{x}_5$
If	$\bar{z}_1z_2z_3$	then	$Y_1 = \bar{x}_1$	$Y_2 = x_2$	$Y_3 = x_3$	$Y_4 = x_4$	$Y_5 = x_5$
If	$z_1z_2z_3$	then	$Y_1 = x_1$	$Y_2 = \bar{x}_2$	$Y_3 = x_3$	$Y_4 = x_4$	$Y_5 = x_5$
If	$\bar{z}_1\bar{z}_2\bar{z}_3$	then	$Y_1 = x_1$	$Y_2 = x_2$	$Y_3 = x_3$	$Y_4 = x_4$	$Y_5 = x_5$

One can conclude from the table that

$$
\begin{aligned}
F_{(Y_1)} &= x_1(z_1\bar{z}_2\bar{z}_3 + \bar{z}_1z_2\bar{z}_3 + \bar{z}_1\bar{z}_2z_3 + z_1z_2z_3 + \bar{z}_1\bar{z}_2\bar{z}_3) + \bar{x}_1\bar{z}_1z_2z_3,\\
F_{(Y_2)} &= x_2(z_1\bar{z}_2\bar{z}_3 + \bar{z}_1z_2\bar{z}_3 + \bar{z}_1\bar{z}_2z_3 + \bar{z}_1z_2z_3 + \bar{z}_1\bar{z}_2\bar{z}_3) + \bar{x}_2z_1z_2z_3,\\
F_{(Y_3)} &= x_3(\bar{z}_1z_2\bar{z}_3 + \bar{z}_1\bar{z}_2z_3 + \bar{z}_1z_2z_3 + z_1z_2z_3 + \bar{z}_1\bar{z}_2\bar{z}_3) + \bar{x}_3z_1\bar{z}_2\bar{z}_3,\\
F_{(Y_4)} &= x_4(z_1\bar{z}_2\bar{z}_3 + \bar{z}_1\bar{z}_2z_3 + \bar{z}_1z_2z_3 + z_1z_2z_3 + \bar{z}_1\bar{z}_2\bar{z}_3) + \bar{x}_4\bar{z}_1z_2\bar{z}_3,\\
F_{(Y_5)} &= x_5(z_1\bar{z}_2\bar{z}_3 + \bar{z}_1z_2\bar{z}_3 + \bar{z}_1z_2z_3 + z_1z_2z_3 + \bar{z}_1\bar{z}_2\bar{z}_3) + \bar{x}_5\bar{z}_1\bar{z}_2z_3.
\end{aligned} \tag{6}
$$

The functional schematic of such a corrector coincides with the one given on Fig. 2.

By eliminating contacts of the Z_i relays on the basis of (5) and (5'), and by using the notation of (1) and (1'), we also arrive at (2) and (2').

The synthesis of the corrector circuit can be executed in the following way. We construct the (1,3)-terminal network which realizes the conductances c_2, c_3, and c_4. Then, by means of diode networks, we realize the function of $F_{(Y_1)}$, $F_{(Y_2)}$, $F_{(Y_3)}$, $F_{(Y_4)}$, and $F_{(Y_5)}$.

The corresponding circuit is shown on Fig. 3.

In the general case, to synthesis a scheme for any n, it is convenient to construct a (1,p)-terminal network† with a diode network instead of a (1,n)-terminal network if, in the system of working signals, there are many in which the sign "1" arises in the same bit positions.

With this, the number of diodes R does not exceed Q:

$$Q = 2[2^p - (p+1)], \tag{7}$$

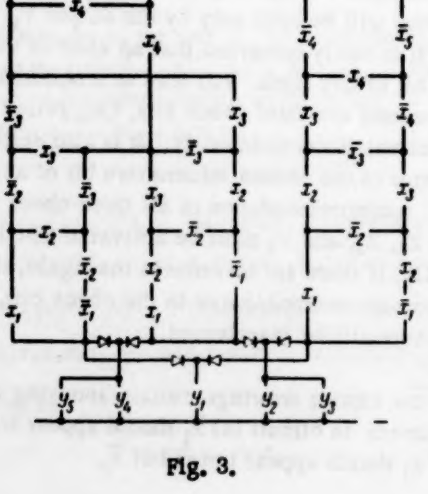

Fig. 3.

whereby R = Q if one of the n functions $F_{(Y_1)}$, $F_{(Y_2)}$, . . ., $F_{(Y_n)}$, for example, $F_{(Y_l)}$, is the sum of the aforementioned p functions, i.e.,

$$F_{(Y_l)} = c_1 + c_2 + \ldots + c_p, \tag{8}$$

and all the remaining functions are all the possible logical consequences of applying (8). (Obviously, in this case, $n - 1 = 2^p - 2$).‡

To convince ourselves of this, we use the method of mathematical induction, first noting that, to construct each new output from two existing ones, two diodes are required (Fig. 4).

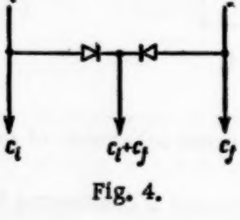

Fig. 4.

We first consider the case $n + 1 = 2^p$. We convince ourselves of the correctness of formula (7) for p = 2 (n = 3). In this case

$$F_{(Y_1)} = c_1 + c_2, \quad F_{(Y_2)} = c_1, \quad F_{(Y_3)} = c_2$$

and

$$Q = 2[2^2 - (2+1)] = 2(4-3) = 2,$$

which is in accord with the actual facts (Fig. 4).

We now assume that (7) is valid for some p > 2, and we prove its validity for p + 1.

† Here, p is the number of different Boolean functions c_1, c_2, . . . , c_p entering into the expressions $F_{(Y_i)}$ (i = 1, 2, . . . , n), each of which is the sum of the constituent units corresponding to some working signal and all of its distorted variants.

‡ We exclude from the number of all the consequences the two trivial ones: 0, and the function $F_{(Y_l)}$ itself.

Let there be given p + 1 outputs, and let it be necessary to obtain from them all possible outputs by interconnecting the primordial p + 1. We assume that, from p outputs,we have already obtained all their possible outputs (their number will be $C_p^1 + C_p^2 + C_p^3 + \ldots + C_p^p$) and that we have used $2[2^p-(p+1)]$ diodes for this. We shall connect the (p+1)th output with each of these outputs, using two diode elements for each of these connections. With this, we shall use

$$2\,[C_p^1 + C_p^2 + C_p^3 + \ldots + C_p^p] = 2^p\,(2^p - 1)$$

diodes.

Consequently, we shall use a total of

$$2\,(2^p - 1) + 2\,[2^p - (p+1)] = 2\,\{2^{p+1} - [(p+1)+1]\}$$

diodes, q.e.d.

If $n + 1 < 2^p$, then the number of connections and the number of diode elements used will be smaller.

It should be mentioned that the assumption that the correctors be realized by relay contact circuits is not an essential one. In [9] there was presented a method of synthesizing correctors from contactless elements by structural formulas of the form of (2) and (2').

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10-CHANNEL AUTOMATIC ELECTRONIC-RELAY OPTIMIZER

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A description is given of a 10-channel automatic electronic-relay optimizer. The operational part of the optimizer consists of electronic operational amplifiers, while the controlling portion is comprised of relays and stepping switches.

The optimizer is designed to operate with electronic analog computers and is used to obtain solutions to complicated variational problems, and problems of finding extrema. An example of problem solution by means of the optimizer is given.

Certain problems in finding extrema cannot be solved analytically, either because analytic expressions of the functions are not known, or because these expressions are too complicated. This is particularly the case for the problems related to the finding of extrema of functions of many variables. In actual systems, or in systems simulated on analog computers, one can, by giving increments to the individual variables, easily determine partial derivatives and, consequently, gradients of a function. By moving either in the direction of the gradient or in a directly opposite direction, one can find a maximum or a minimum of a function. The processes of automatically determining partial derivatives and of moving in the direction of the gradient can be implemented by means of the so-called automatic optimization systems. If, from the conditions of the problem, certain boundaries are imposed on the variables then, as was shown in [1], the determination of an extremum lying on a boundary is possible in this case.

The theoretical considerations bearing on the automatic search process were presented in [1, 2]. A complete electronic variant of an optimizer was described in [3].

Below there is given a description of an electronic-relay instrument which executes automatic search for extrema. For brevity, we shall henceforth call this instrument an optimizer. Participants in the design and adjustment of the optimizer were I. N. Bocharov, A. V. Kalinina, R. I. Stakhovskii, and the author of the present paper.

The optimizer permits one to find the extremum of a function of many variables (up to 10) with the presence of limitations (boundaries).

The function whose extremum is to be sought, as well as the function which is bounded, are applied to the optimizer in the form of dc voltages whose range of variation is ±100 v. The optimizer can operate with objects which continuously output the value of the function whose extremum is being sought, as well as with objects which output the function value only at certain moments of time (for example, after solution on an electronic analog computer).

The optimizer can be divided into two parts: the operational part and the controlling part. The basic computing portions of the automatic optimizer are the electronic analog blocks. The output voltages $x_1, x_2, \ldots, x_{10}$ are formed on dc integrators; the range of variation of the output voltages is ± 100 v. The size of the trial step, applied for the determination of the partial derivatives, is regulated within the limits of 0 to 15 v. The choices just cited are related to the circumstance that the optimizer is basically designed to operate with electronic analog computers and, with such choices for the computing (operational) part, matching of the optimizer with analog computers is executed with extreme simplicity. The accuracy of the operations effected by the computer and the optimizer is of the same order of magnitude in all cases. The accuracy with which the extremum is determined depends on the character of the problem to be solved. For a test object, which was the sum of the moduli of the ten variables $Q = k\sum_{i=1}^{10} |x_i|$, the maximum relative error in determining Q_{extr} was 1%.

The optimizer's controlling portion analyzes the data applied to the optimizer and analyzes the state of the optimizer itself and, as a function of the information obtained, carries out some switching or other. Relays and stepping switches are the basic elements of the controlling portion. Adequate operation of such a controlling block is possible for frequencies of the synchronizing pulses up to 5 cps, i.e., it requires about 0.2 sec to apply the trial increments and to remember (store) the partial derivatives. Such a speed of operation is completely sufficient for operation with ordinary electronic computers which are also relay-controlled.

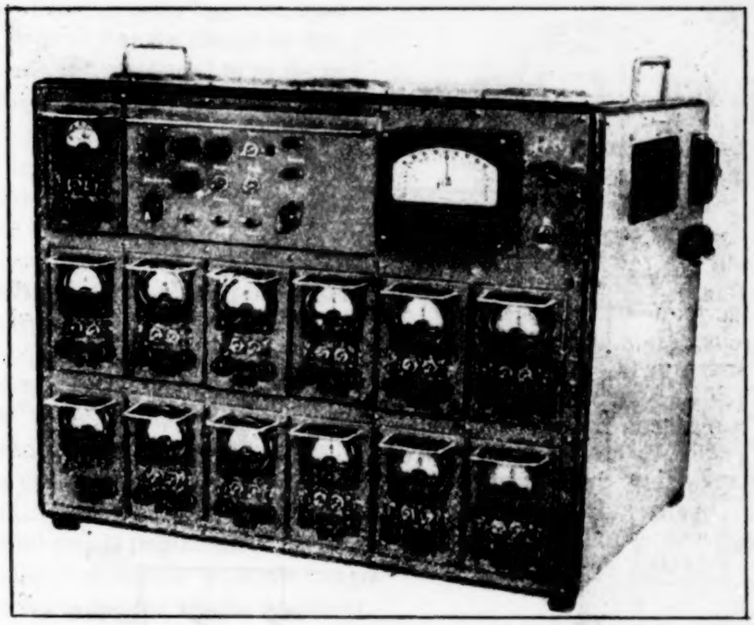

Fig. 1.

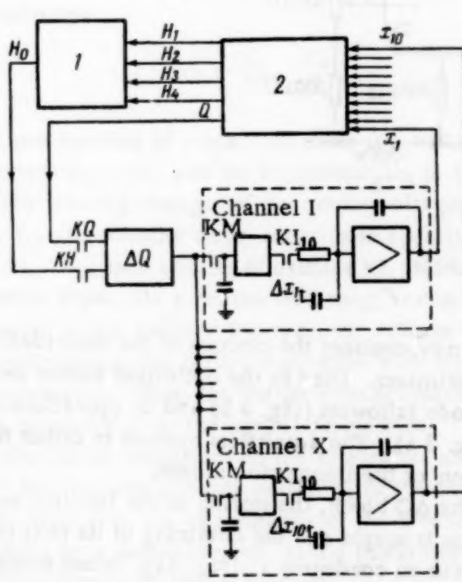

Fig. 2. 1 is the block for forming limitations, 2 is the object.

An exterior view of the optimizer is shown on Fig. 1. The optimizer design incorporates, in one chassis, the controlling block, the limitations block, the ΔQ block, the test block and 10 channel blocks. The front panel of the optimizer is comprised of the front panels of the individual blocks, on which are provided the necessary devices for manual control and testing of the instrument. The dimensions of the optimizer, without the power supply, are 650 × 500 × 450 cm.

Operational Portion

Figure 2 shows the functional schematic of the optimizer's operational portion. The theory of operation amounts to the following: the quantity Q, whose extremum is sought, is applied to the ΔQ block. This block remembers the function and determines its variation. We assume, that, in the ΔQ block, there has been remembered the value of the function for some values of $x_1, x_2, \ldots, x_{10}$; if we now give a trial increment Δx_{1t} to the first variable then, at the output of the ΔQ block we obtain the increment of the function $\Delta Q_1 \approx (\partial Q/\partial x_1)\Delta x_{1t}$, i.e., a quantity which is proportional to the partial derivative of the function with respect to the first variable. Simultaneously with the application of the trial increment, contact KM_1 closes, and storage of the quantities $\pm(\partial Q/\partial x_1)\Delta x_{1t}$ occurs in the memory link of channel 1.

After this, a trial increment is applied to the second variable, etc., up to the tenth variable. At the end of the cycle, there have been stored in the channels' memory links the quantities which are proportional to the partial derivatives, or components of the function's gradient. Upon simultaneous closure, at time t_s, of contacts KI_1, KI_2, . . . , KI_{10}, the integrators receive the increments

$$\Delta x_{1r} = \pm \frac{\partial Q}{\partial x_1} \Delta x_{1t} k t_s ,$$

$$\Delta x_{2r} = \pm \frac{\partial Q}{\partial x_2} \Delta x_2 \, k t_s \quad \text{etc.}$$

The trial increments Δx_t and the integration constants $\underline{k}$ are identical in all the channels, so that the operating variations of the variables (the working steps) are proportional to the components of the gradient. After a working step, one can repeat the determination of the gradient and carry out a new working step along the new gradient; in this case, the search is carried out by the gradient method. After the working step one can determine the sign of the function's change and, if this change

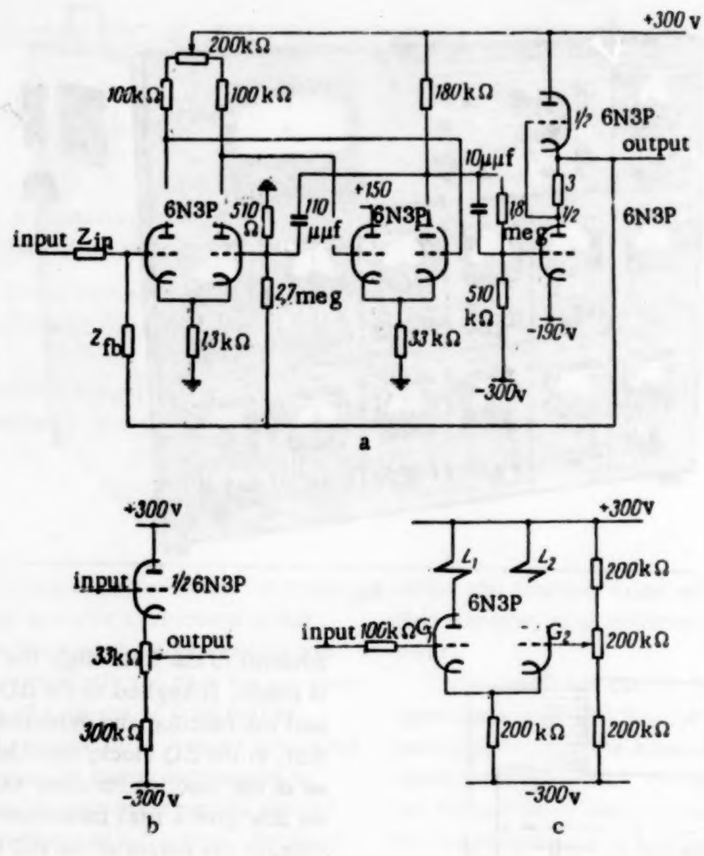

Fig. 3.

occurred in the proper direction, then the working step along the previous gradient may be repeated until such time as the increment changes sign, i.e., until such time as a partial extremum in the direction of the previous gradient is reached. This method of search is called the method of steepest descent. There are a number of switches on the controlling block whose positions correspond to the various search methods.

If limitations are imposed on the system's coordinates i.e., if some function $H_0(x_1, x_2, \ldots, x_{10})$ must not exceed a given value, then the search for an extremum is executed in the following manner. Optimization of the function Q proceeds until the function H_0 goes beyond the limits of the admissible region; with this, there is automatically switched to the input of the ΔQ block, not the function Q whose extremum is sought, but the bounded function H_0. In this case, $x_1, x_2, \ldots, x_{10}$ are so changed that they move out of the region of limitation. After the region of limitation has been left, the process of finding an extremum of function Q is continued. If an extremum is found either on the boundary or inside the region of limitation, there then occurs an oscillation about the boundary and, in the process of these oscillations, the optimizer finds the true extremum or a conditional extremum on the boundary.

We now consider the circuits of the individual blocks of the optimizer. Used in the optimizer blocks are standard cathode followers (Fig. 3 b) and dc operational amplifiers (Fig. 3 a). The amplifiers operate in either the amplification or the integration modes.

In the ΔQ block, the storing of the function whose extremum is sought and the obtaining of its next increment occur on condenser C_1 (Fig. 4a). When contact R_m is closed, the condenser charges up to the potential Q_0 while, when the contact is opened and Q varies, the following voltage is formed at the output of the cathode followers:

$$\Delta Q = (Q - Q_0)\, k_{tc},$$

where k_{tc} is the cathode follower's transfer coefficient.

The function increment obtained from the cathode follower is amplified and applied to the channels' memory links. There are inverters at the input of the ΔQ block which are switched in to the input Q by toggle switch T_{QI} and to input H_0 by toggle switch T_{HI}. This makes it possible to change the signs of the partial derivatives $(\partial Q/\partial x_i)\Delta x_{it}$ and $(\partial H_0/\partial x_i)\Delta x_{it}$, and thereby to carry out either minimization or maximization.

Figure 4b gives the functional schematic for one channel. Storage of the partial derivative occurs on con-

denser C_2 upon operation of storage relay R_m, i.e., upon closure of contact KM. In order that the charge on the condenser not be decreased when connected in to the integrator's input, the voltage is taken off from it by means of the cathode follower. The cathode follower is based on tube 6 N 3P which has low grid current; the change in the condenser's charge due to grid current does not exceed 0.1 v/ min. To eliminate potential leakage from condensers C_2 and C_3 (and also from condenser C_1 in the ΔQ block) as well as a fatigue effect, polystyrene MPGT condensers of 1 μf capacity are used in the optimizer.

The output voltages $x_1, x_2, \ldots, x_{10}$ are supplied from the dc amplifier-integrators. The amplifier has high gain, uses tube 6N3P in its first stage, and the operating mode of its first stage is chosen so as to minimize grid current. Integrating capacitor C_3 is connected in the amplifier's feedback path.

Application of the trial step is implemented by means of condenser C_4. Upon closure of contact RR at the integrator input, a step voltage U_{ts} is applied via the condenser. As is known, the integrator is an amplifier under these conditions, so that the following increment appears at the integrator output.

$$\Delta x_t = U_{ts} \frac{C_4}{C_3}.$$

Upon opening of contact RR there is a voltage step of the opposite sign, and the increment Δx_t is tapped off.

The working changes of the output voltages $x_1, x_2, \ldots, x_{10}$ (the working steps) occur upon operation of step relay R_s, i.e., upon closure of contact KI. With this, the integrator input, via a resistor of 1 meg, is connected to the output of the cathode follower, the voltage on which is proportional to the partial derivative with respect to these variables. Thus, the output voltages receive increments proportional to the partial derivatives.

The initial conditions on the integrators are established by connecting toggle switch T_2 to the channel block. With this, the capacitance of the feedback path is shunted by a resistance and, at the integrator's input, a voltage from a divider is applied.

The circuit for forming the limitation H_0 is given in Fig. 4c. The circuit includes an amplifier with four inputs. There is a vacuum-tube diode at each input. Thus, the circuit gives the sum of only the positive voltages H_i which is just the bounded function H_0. The function to be limited is applied to the limitation indication circuit in the indicator block.

The circuit for indicating limitiations (similar to the circuit used for indicating partial minima and indicating the final solution) is shown on Fig. 3c. The circuit works as follows. In the differential stage's anode circuit there are connected the windings of the polarized relay RP-4. The windings are connected toward each other. On one of the grids, G_2, the constant voltage U_0 is applied, while on the second grid, G_1, the voltage U to be compared is applied. If $U > U_0$, then the current in winding L_1 is

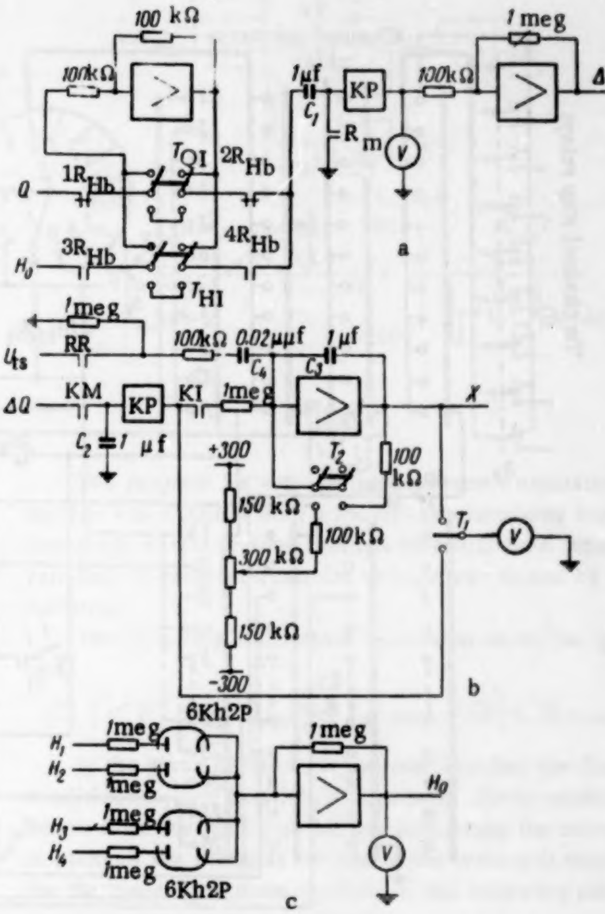

Fig. 4.

greater than that in winding L_2, and the polarized relay is found in one of its positions. For $U < U_0$, the relay is transferred to its other position. Such a circuit allows one to fix the transition of a voltage through any threshold level with an accuracy of about 0.5 v; the hysteresis of the circuit is less than 0.1 v.

Controlling Block

The controlling block is comprised of a stepping switches, relays and switches of various types of operation (Fig. 5).

The first stepping switch SSI executes successive switching in of the memory relays of the channels and, by the Gauss-Seidel method, also executes the individual switching of the channels' step relays. Simultaneously with the switching of the channels' memory relays, the trial increments for the variables are supplied. Thus, as SSI moves along commutator segments 2-11, the gradient is determined for the function whose extremum is being sought. When on the first commutator segment, SSI applies voltage to SSII, causing execution of the operations of working-step implementation and determination of the change in the function during the working steps. With the Gauss-Seidel method, these operations are possible on seg-

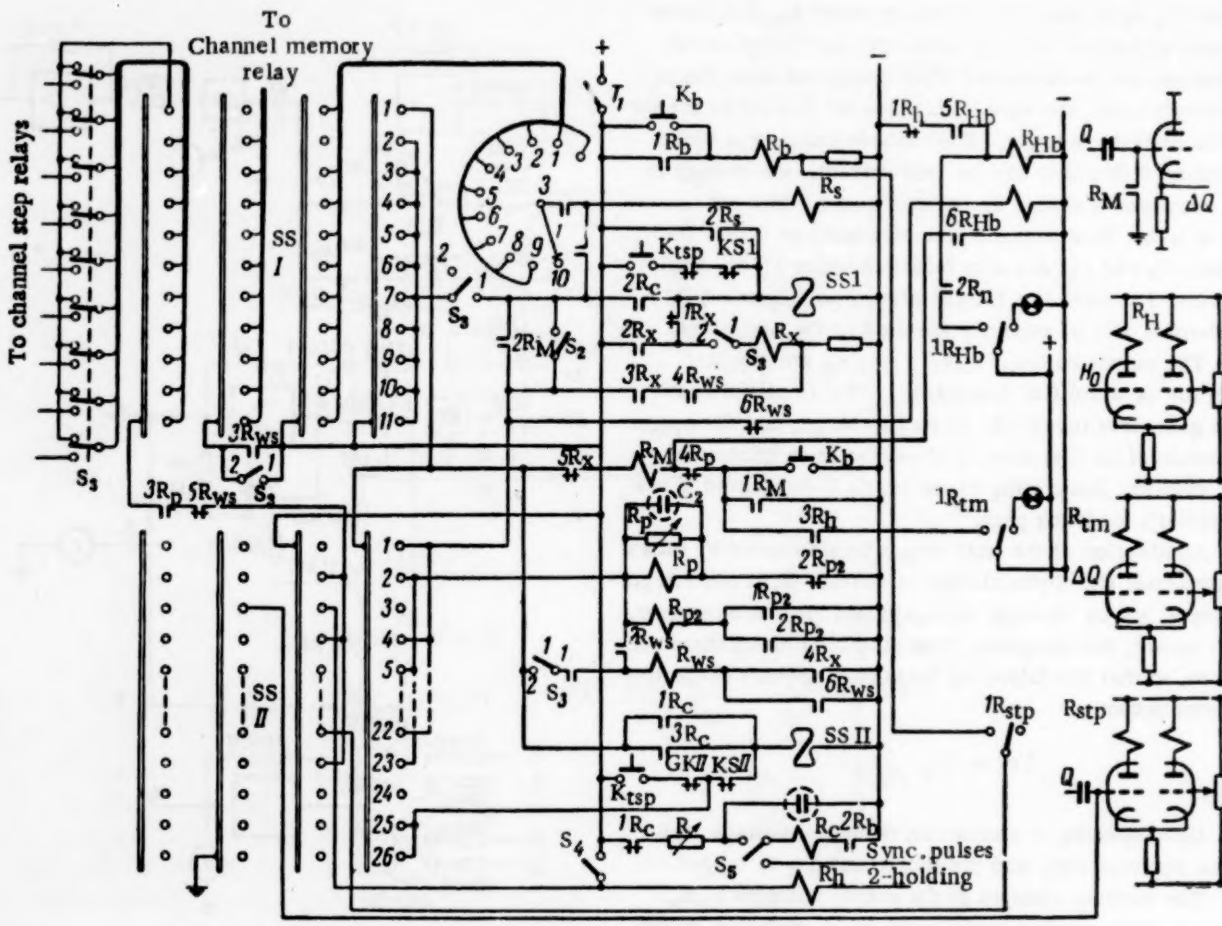

Fig. 5.

ments 2-11, while commutator segment of stepping switch SSI is passed over by shunt running.

Various relays implement the following basic functions:

1. Start relay R_b, when the start button is pushed, switches in cycle relay R_c.

2. Cycle relay R_c operates upon the reception of a sync pulse from the synchronization block, and transfers SSI or SSII to the following commutator segment, i.e., implements transfer from one operation to the following one. For the optimization of continuous objects, the relay is transferred to the autooscillatory mode by toggle switch S_5.

3. Relay R_s, for shortening the variables cycle, switches SSI to shunt running in order to pass rapidly over the commutator segments which correspond to unused channels.

4. Step relays R_p and R_{p2} provide the required delay to the working step. This delay is determined by condenser C_2 and resistor R_s.

5. The minimum relay R_M operates when a partial (relative) minimum is reached. After the storage operation, relay R_M, by its own contact $2R_M$, switches SSI to gradient determination. The operation of gradient determination must also be executed when the instrument is started and upon input or output of limitations. In the first case, relay R_M is switched on by a normally open contact of the start button and, in the second case, by a special contact $6R_{Hb}$ of limitation blocking (holding) relay R_{Hb}. This contact is closed only in the periods when the relay is transferring from one state to another.

6. Holding relay R_h operates after the termination of a working step. At this moment there is implemented the analysis of the increment of the function whose extrumum is being sought, contact $3R_h$ is closed and the minimum relay is switched to the circuit for minimum indication.

Contact $1R_h$ releases relay R_{Hb}, and contact $2R_h$ switches the limitation holding relay to the limitation indication circuit. After release of R_h, the state of relay R_{Hb} depends on the state of relay R_H at the moment of holding, and remains unchanged until the following operation of relay R_h.

For the optimization of functions obtained after a solution on an analog computer, it is necessary that the holding relay operate at the moment when there is a value of the function to be optimized available at the computer. For this, 2-holding is used.

For an extremum search using the Gauss-Seidel method, the memory relays in the channels are connected-in via the commutator segments of SSI and contact $3R_{ws}$

of the working step relay. To switch this relay in, there is auxiliary relay R_x. Relay R_x operates simultaneously with SSI. Thus, when SSI transfers to the following segment, i.e., when the optimization cycle for the previous variable terminates, relay R_x is found in the operated state. Since contact $4R_x$ is closed, then simultaneously with the operation of R_p working step relay R_{ws} operates and, by its contact $3R_{ws}$, it connects in the memory relay in the corresponding channel; the operation of partial derivative determination then ensues. Contact $5R_{ws}$ cuts out the channel working step relay. Simultaneously relay R_{ws}, by contact $4R_{ws}$, cuts out relay R_x. At the time of the following working step, contact $4R_x$ will be open, relay R_{ws} will not operate, so that a working step will be executed, etc. With the Gauss-Seidel method, the channels' working step relays are opened by switch S_3, and the working step is executed for one of the channels.

The switches have the following basic functions: T_1 applies voltage to the circuit; the closed position of switch S_2 corresponds to search by the gradient method, i.e., after each working step the determination of a new gradient is made; S_3 switches of optimizer from operation by the gradient method to operation by the Gauss-Seidel method; S_4 switches in 2-holding (secondary holding) for discrete objects; S_5 connects the cycle relay to a source of external pulses; S_6 reduces the number of variables for optimization.

Solution of a Problem by Means of the Automatic Optimizer

In addition to the solution of problems without limitations, such as the automatic approximation of a given curve [3], the optimizer permits the solution of complicated problems with limitations.

The problem arose of finding the optimal control program for a motor-generator system such that, during a given time τ, the motor would make the maximum number of rotations from the moment of starting to the moment of stopping, while observing the limitations imposed on the system's internal coordinates.

As the first step, it was assumed that the motor's excitation current remains constant. An approximate linearized system of equations for the object has the form:

$$T_1 \frac{dE_g}{dt} + E_g = kU_1,$$

$$E_g - c\Omega I_{em} = I_a R_a, \quad \frac{dX}{dt} = \xi_0 \Omega.$$

Here, U_1 is the voltage applied to the generator's excitation winding, E_g is the generator's emf, I_a is the motor's armature current, I_{em} is the motor's excitation current, Ω is the angular velocity of the motor shaft, X is the angle of rotation of the output shaft, connected by a reducer to the motor shaft, R_a is the resistor of the circuit of the motor armature and the generator armature.

The circuit set up on the analog computer to simulate the given differential equations is shown on Fig. 6a.

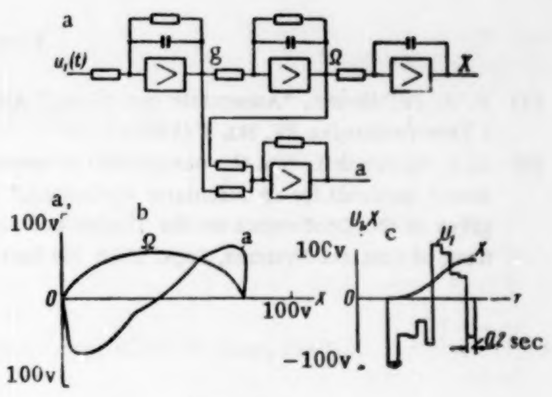

Fig. 6.

The program for varying the generator's excitation voltage was supplied from a special synchronizing block. The program was divided into ten identical time intervals and, in each interval, the voltage was chosen by the optimizer.

The following limitations were imposed on the system:

$$|U_1| \leqslant U_{1\,\max} \quad |I_a| \leqslant I_{a\,\max} \quad |\Omega| \leqslant \Omega_{\max}.$$

In the given problem, it was required that the final conditions $\Omega_{end} = 0$ and $E_g = 0$ be met. These conditions derive from the posing of the problem, since the complete stopping of the motor at the end of the process is required. For the meeting of these conditions, the following additional limitations were imposed:

$$\Omega_{end} \leqslant 0 \text{ and } E_{g\,end} \leqslant 0.$$

After starting of the system from null initial conditions, the optimizer automatically found the optimal program. In the majority of cases, the maximum X lay at the point of intersection of the boundaries of limitation:

$$\Omega_{end} \leqslant 0 \quad E_{g\,end} \leqslant 0, \quad I_a \leqslant I_{a\,\max}.$$

The optimal processes found by the optimizer are shown on Fig. 6, b and c.

CONCLUSIONS

The automatic multichannel optimizer is an instrument allowing one to solve variational problems and problems of finding extrema of functions of many variables.

The given optimizer was designed basically to operate with electronic analog computers with relay control. However, the principles on which the instrument is based can be used for creating optimizers to operate with actual objects. It is possible to have optimization of actual objects carried out in two time scales, i.e., rapid finding of extrema on an analog computer and a subsequent transfer of the results to the actual object.

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* See English translation.

RELIABILITY CHARACTERISTICS OF RESISTORS AND CONDENSERS

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Methods are considered for the presentation of reliability characteristics of resistors and condensers as functions of ambient air temperature and power dissipation or applied voltage.

The reliability of constant carbon resistors, among the most widely used elements of different systems, depends on the physicochemical processes occurring both within the conducting layer and on its surface. Therefore, the reliability of resistor operation is determined, to a large degree, by the choice of materials, the design, and the technology of manufacture. In addition, the reliability of resistor operation also depends on the correctness of the selection of its operating regimen and on external conditions.

In designing circuits containing resistors and condensers, one must know how to determine their reliability under given conditions of use, which may be characterized by the ambient temperature θ_X, dissipated power P_X, voltage U_X and length of serviceable life T. For this end, it is convenient to introduce characteristics which give the connections between reliability and the parameters which define the conditions of use. The consideration of these connections comprises the subject matter of the present paper.

1. By considering the breakdowns of resistors to be random events, we have that the number of operational failures during time interval dt will be the larger, the greater is the number N of resistors, i.e.,

$$-dN = cNdt. \qquad (1)$$

It follows from this that

$$\ln N = -ct + \ln k \quad \text{or} \quad N = ke^{-ct}.$$

Since $N = N_0 = k$ for $t = 0$, then

$$N = N_0 e^{-ct} \quad \text{or} \quad \frac{N}{N_0} = p = e^{-ct}.$$

Here, N is the number of existing elements (resistors) at time $\underline{t}$: $N/N_0=p$ is the reliability of action of the set of N_0 elements at time $\underline{t}$; $c = 1/\tau$ is the reliability factor (the relative speed of breakdown).

Experimental data substantiate the foregoing dependence of element reliability on time for the fundamental portion of the characteristic $p = N/N_0 = f(t)$ (segment AB on Fig. 1). Deviations occur only in the initial period, when the unconditioned elements go out of order, and after point B when, due to accelerated aging and to other irreversible changes, the element breakdown rate begins to accelerate.

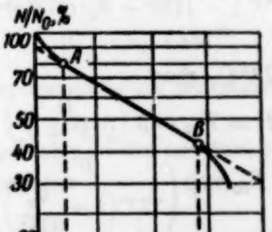

Fig. 1

Point B determines the length of serviceable life T for the given conditions of operation.

We now consider the dependence of the coefficient $\underline{c}$ on the resistor's heating temperature θ_X.

The velocity of change of material during the reaction process is defined by the equation

$$-\frac{dn}{dt} = \alpha n,$$

where $\underline{n}$ is the quantity of material at the moment of time under consideration and α is the constant velocity of the reaction.

From the equation given it follows that

$$\frac{dn}{n} = -\alpha\, dt,$$

i.e.,

$$\ln n = -\int_0^T \alpha\, dt,$$

or, for α = const,

$$\ln \frac{n}{n_0} = -\alpha T.$$

The dependence of the constant reaction speed on temperature is determined, according to Arrhenius, by the formula

$$\alpha = A_k \exp\left(-\frac{B}{273° + \theta_x}\right),$$

where A_k and B are constants and $(273 + \theta_X)$ degrees is

the absolute temperature.

Consequently,

$$\ln\frac{n}{n_0} = -A_k \exp\left(-\frac{B}{273^\circ + \theta_x}\right)T.$$

We find from the last expression that

$$T = \left[\frac{1}{A_k}\ln\frac{n_0}{n}\right]\exp\left(\frac{B}{273^\circ + \theta_x}\right)$$

or

$$T = T_\infty \exp\left(\frac{B}{273^\circ + \theta_x}\right).$$

If $\theta_x \lesseqgtr 273^\circ$ C, then

$$\frac{T}{T_0} = \frac{T}{T_\infty}\,\frac{\exp\left(\frac{B}{273^\circ + \theta_x}\right)}{\exp\left(\frac{B}{273^\circ + \theta_0}\right)} =$$

$$= \exp\left(-\frac{B(\theta_x - \theta_0)}{(273^\circ + \theta_x)(273^\circ + \theta_0)}.\right)$$

If $\theta_x < 273^\circ$ C, then

$$T = T_\infty \exp\left(\frac{B}{273^\circ + \theta_x}\right) =$$

$$= T_\infty \exp\left(\frac{B}{273^\circ\left(1 + \frac{\theta_x}{273^\circ}\right)}\right) \approx$$

$$\approx T_\infty \exp\frac{B}{273^\circ}\exp\left(-\frac{B}{273^2}\theta\right) =$$

$$= T_a \exp(-\beta\theta_x),$$

or

$$T = T_0 \exp[-\beta(\theta_x - \theta_0)],$$

$$T_a = T_\infty \exp\frac{B}{273^\circ},$$

$$\beta = \frac{B}{273^2},\ T_0 = T_a \exp(-\beta\theta_0).$$

By denoting by s_0 the amount of material in one resistor, we get that $n = s_0N$ and, consequently $-s_0 dN/dt = s_0 N\alpha$ or $-dN/dt = N\alpha$.

By comparing the last equation with Eq. (1), we get that $\alpha = c$. We denote by c_0 the value of $\underline{c}$ which corresponds to θ_0; we then find that

$$\frac{c}{c_0} = \frac{A_k \exp\left(-\frac{B}{273^\circ + \theta_x}\right)}{A_k \exp\left(-\frac{B}{273^\circ + \theta_0}\right)} =$$

$$= \exp\left(\frac{B(\theta_x - \theta_0)}{(273^\circ + \theta_x)(273^\circ + \theta_0)}\right) \qquad (2)$$

The value of the temperature θ_x is determined as the sum

$$\theta_x = \theta_{0x} + R_t P_x, \qquad (3)$$

where θ_{0x} is the ambient temperature, R_t is the thermal impedance for heat transfer from the surface to the ambient medium and P_x is the power dissipated in the electrical impedance.

By substituting from (3) to (2), and by denoting by c_N the value of $\underline{c}$ for the nominal load P_N and the admissible temperature $\theta_{0x} = \theta_{0N}$, we get

$$\frac{c}{c_N} = \exp\frac{B\left[\theta_{0x} - \theta_{c_N} + R_t P\left(\frac{P_x}{P_N} - 1\right)\right]}{(273^\circ + \theta_{0x} + R_t P_x)(273^\circ + \theta_{0_N} + R_t P_N)}.$$

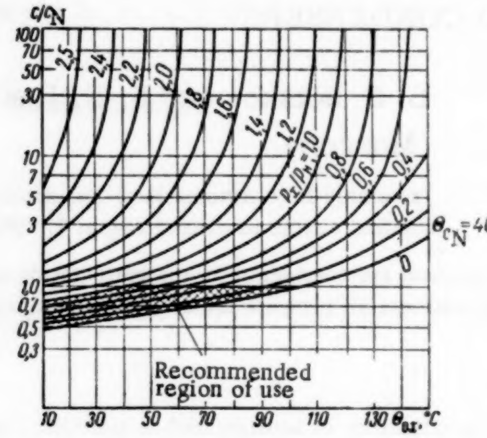

Fig. 2

Figure 2 shows the function c/c_N for one type of resistor. This function is also the characteristic $c/c_N = f(\theta_{0x}, P_x)$ for the resistor.

The quantities B, R_t and c_N are obtained by processing experimental data. The value θ_{0N} corresponds to the maximum permissible temperature for the nominal load $P_x = P_N$.

It should be kept in mind that the values of B, R_t, and c_N vary somewhat as functions of the type and the nominal resistance. However, for determining resistor reliability in the process of designing a scheme, it suffices to estimate the average value of reliability for a given type of resistor. Such averaged values of c_N and B are given in the following table.

If the limiting admissible value for the resistor temperature is $\theta_{x\,max}$, then the magnitude of the admissible power dissipation equals $P_x = (\theta_{x\,max} - \theta_{0x})/R_t$.

Figure 3 shows the function $P_x = f(\theta_x)$.

The value of $\theta_{0x} = \theta_{nom} = \theta_{xN}$ is chosen from the condition limiting the relative variation of the resistance $\Delta R/R = f(\theta_x)$ for $\theta_{0x} = \theta_{xN}$, and should be a value which is achievable technologically or which is determined from the given conditions of use.

The method of choosing the value of $P_{nom} = P_N$ consists of this: that knowing the length of useful life T_0 and the temperature θ_0 corresponding to it, we give a value to the admissible service life T. Then, based on the foregoing, we obtain

$$P_N = \frac{1}{R_t}\left[\frac{B}{\frac{B}{273^\circ + \theta_0} + \ln\frac{T}{T_0}} - (273^\circ + \theta_{0x})\right].$$

With the condition that the value of $\underline{c}$ remain invariant, the limiting value $\theta_{0x} = \theta_{x\,max}$ is determined if we set $P_x = 0$, i.e.,

$$\theta_{x\,max} = \theta_{0x} = \frac{B}{\frac{B}{273^\circ + \theta_0} + \ln\frac{T}{T_0}} - 273^\circ.$$

Type of resistor	Nominal power in W	Admissible limiting temperature θ_{max}, in °C	Limiting ambient temperature θ (for P = P_N), in °C	Thermal impedance R_t, in °C/w	c_k (mean value), hr^{-1}	β
VS — 0.12 (ULM — 0.12)	0.12	130	180	350		
VS — 0.25	0.25	130	95	140		
VS — 0.5	0.5	130	85	90	$1.0\cdot10^{-8}$	0.13—0.18
VS — 1.0	1.0	130	65	65		
VS — 2.0	2.0	130	70	45		
VS — 5.0	5.0	130	35	19		
VS — 10.0	10.0	130	50	8		
VS per spec.	0,12—10,0	110	40	70		
MLT — 0.5	0.5	130	70	50		
MLT — 1.0	1.0	130	—	—	$1.7\cdot10^{-8}$	0.11—0.12
MLT — 2.0	2.0	130	—	—		

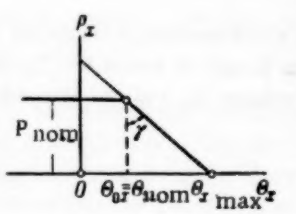

Fig. 3

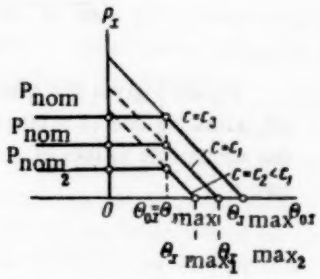

Fig. 4

Conversely, by giving various values: $c = c_1$, $c = c_2$, $c = c_3, \ldots$, we obtain various magnitudes of $\theta_{x\ max}$. This makes it possible to define P_N for a given θ_N = const (Fig. 4). The graph of Fig. 4 is another form of the same characteristic $c/c_0 = f(\theta_{0x}, P_x)$ which relates the change of reliability to the change of temperature of the ambient medium and to the change of power dissipation P_x.

The value of the thermal impedance can be found from the formula

$$R_t = \frac{1}{\mu S + 2\lambda S_0 / l_0},$$

where μ is the total heat transfer coefficient, S is the cooling surface, λ is the thermal conductivity of the current leads and S_0 and l_0 are their cross section and length. The expression for the heat transfer coefficient is written in the form

$$\mu = \mu_1 + b\,\frac{(273° + \theta_x)^4 - (273° + \theta_{0x})^4}{\theta_x - \theta_{0x}},$$

where μ_1 is the heat transfer coefficient due to gas heat conduction and convection. It should be kept in mind that the value of μ_1 varies with changes in p_{h_x}, the pressure of the air surrounding the resistor, by the law

$$\mu_{1h_x} = \mu_{1h_0}\sqrt{\frac{p_{h_x}}{p_{h_0}}},$$

where p_{h_0} is air pressure at sea level. The magnitude of μ_1 also depends on the temperature difference $\theta_x - \theta_{x_0} = \Delta\theta$. The relationship can be expressed approximately by the formula

$$\mu_1 = \mu_{10}(\Delta\theta)^m,$$

where μ_{10} corresponds to the temperature difference $\Delta\theta = 1$°C.

Thus, the value of the nominal power is determined from the expression

$$P_N' = (\theta_{x\,max} - \theta_N)\left\{S\left[\mu_{10}(\Delta\theta)^m\sqrt{\frac{p_{h_x}}{p_{h_0}}} + b\,\frac{(273° + \theta_{x\,max})^4 - (273° + \theta_N)^4}{\theta_{x\,max} - \theta_N}\right] + 2\lambda\frac{S_0}{l_0}\right\}.$$

It follows from this expression that the value of the nominal power must vary somewhat as a function of the air pressure and the value of $\theta_N = \theta_{x\,max} - \theta_N$. It will also depend on the mounting, since l_0 and μ_{10} also vary with this.

The foregoing table gave the values of R_t and c_N for a number of standard resistors, and also gave the permissible temperature of the ambient medium, the value of which changes for resistors of different power ratings.

2. In the case of condensers, it is necessary in estimating the reliability of the finished articles, to take into account both the effect of temperature and the effect of voltage.

The useful life of paper condensers, determined by the aging of the insulation, as a function of the temperature θ_x, is expressed by the formula

$$T_{x0} = T_0 \exp\left[-\beta_n(\theta_x - \theta_0)\right], \qquad (4)$$

where β_n is a coefficient which depends on the form of the insulation.

On the other hand, for dielectric aging in the presence of an electric field, we have the empirical relationship

$$\varepsilon = AT^{-n} = \frac{A}{T^n}, \qquad (5)$$

where ϵ is the electric field strength, T is the length of useful life and A is a constant.

By taking into account that $U = \epsilon d$ we get that $U = AdT^{-n}$, where $\underline{d}$ is the thickness of the dielectric. For condensers with paper insulation, $n \approx 0.2$.

For two different voltages U_x and U_{x0} we have that

$$\left(\frac{U_x}{U_{x0}}\right)^{1/n} = \frac{T_{x0}}{T_x}.$$

In order to take into account the influence both of U_x and of θ_x, we write the ratio T_{x0}/T_x in the form

$$\frac{T_{x0}}{T_x} = \frac{T_0}{T_x}\frac{T_{x0}}{T_0}.$$

Then,

$$\left(\frac{U_x}{U_{x0}}\right)^{1/n} = \frac{T_0}{T_x}\exp\left[-\beta_n(\theta_x - \theta_0)\right],$$

i.e.,

$$\ln\frac{U_x}{U_{x0}} = -n\ln\frac{T_x}{T_0} - n\beta_n(\theta_x - \theta_0). \qquad (6)$$

If the number of condensers remaining in service for U_x = const and θ_x = const is defined by the relationship

$$p = \frac{N}{N_0} = \exp(-c_k T_x)$$

or

$$\ln p = \ln\frac{N}{N_0} = -c_k T_x,$$

then, by giving some value of $\underline{p}$, one can easily determine the value of length of useful life which corresponds to it:

$$T_x = -\frac{\ln p}{c_k}. \qquad (7)$$

By substituting the value of T_x in (6), we get

$$\ln\frac{U_x}{U_{x0}} = -n\ln\left[-\frac{\ln p}{c_k T_0}\right] - n\beta_n(\theta_x - \theta_0). \qquad (8)$$

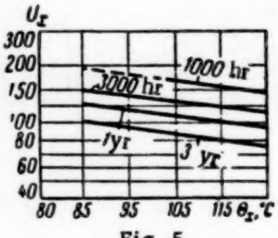

Fig. 5

Figure 5 gives the given characteristic, defined by (6), allowing one to relate the length of useful life T_x to the admissible values of the voltage U_x and the temperature θ_x.

ON THE DESIGN OF MAGNETIC AMPLIFIERS SINGLE-PHASE AND THREE-PHASE SEMICONDUCTOR RECTIFIERS ON AN ACTIVE LOAD ANDOPERATING WITH A COUNTER-EMF

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Methods are presented for the transfiguration of single-phase and three-phase rectifier bridges to equivalent loads, and graphicoanalytic methods are considered for determining the characteristics of saturated reactor magnetic amplifiers operating on active loads and counter-emf's.

Circuits for automatic current stabilizers in charging devices are given, and an example of determining a magnetic amplifier's basic parameters is considered.

Semiconductor rectifiers, in conjunction with magnetic amplifiers, allow the construction of compact automatic control devices. Such devices are widely used for controlling the speed of a motor's rotation by varying the voltage applied to the armature, for charging storage batteries, etc. [1-4]. Setups with magnetic amplifiers and semiconductor rectifiers are increasingly superceding electronic and ionic tubes with grid control.

The currents and voltages at the input and output of rectifier circuits are different, in the majority of cases. The relationships between them vary with changes in the circuit's operating mode (sinusoidal) voltage or sinusoidal current at the rectifier input, etc.) and also with changes in the form of the load (active load, inductive load, counter-emf, etc.). It is usually quite difficult to find the relationships between the input and output currents and voltages, as well as to design the magnetic amplifiers operating in such circuits.

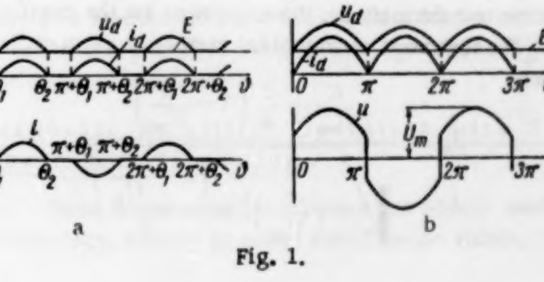
In this paper we present graphicoanalytic methods for determining the characteristics of saturated reactor magnetic amplifiers which operate with single-phase and three-phase semiconductor rectifiers, these methods being based on the transfiguration of the rectifier scheme to a load which is equivalent to it. All other rectifier schemes can be dealt with similarly to the schemes to be considered here.

Magnetic Amplifier Operation with Single-Phase Rectifier Bridge

With operation of a magnetic amplifier on an active load, supplied with rectified current, the instantaneous, average and effective currents at the input and output of the rectifier bridge are all equal to one another, as are the corresponding values of voltage. These relationships are considered for the case of ideal rectifiers, for the absolute magnitudes of the instantaneous values and the average values over a half-period. If the voltage at the rectifier inputs is sinusoidal, then the relationship between the voltage's average value and effective value, as well as the relationship between the analogous current quantities, are constant and equal to 0.9. Therefore, the design of such a circuit engenders no difficulty.

Significant difficulty is engendered in the design of a circuit for rectifier operation on a counter-emf since, in this case, the relationships between the mean and effective values of current and voltage differ from each other, and do not remain constant under variations of the circuit's mode of operation (for example, for sinusoidal voltage and sinusoidal current on the rectifier inputs).

Figure 1 shows the variation of voltages and currents for the operation of a single-phase rectifier bridge on a counter emf for the modes of sinusoidal voltage (Fig. 1a) and sinusoidal current (Fig. 1b) on the rectifier inputs. Here, u and i are the instantaneous values of voltage and current on the rectifier inputs, u_d is the instantaneous value of voltage at the rectifier bridge's output and E is the value of the counter-emf.



a b

Fig. 1.

In the mode of sinusoidal voltageon the rectifier inputs, the angles θ_1 and θ_2 are defined by the following relationships [1]:

$$\sin\theta_1 = \frac{E}{U_m}, \quad \theta_2 = \pi - \theta_1, \tag{1}$$

where U_m is the amplitude of the voltage applied to the rectifier bridge.

We denote by β_i the ratio of the mean value of rectified current I_d to the effective value of current I at the input of the rectifier bridge, i.e.,

$$\beta_i = \frac{I_d}{I}, \tag{2}$$

and, by β_u, the analogous relationship for voltage, i.e.,

$$\beta_u = \frac{U_d}{U}. \tag{3}$$

With a sinusoidal voltage at the input of a single-phase rectifier bridge, the coefficients β_i and β_u have the following values [1]:

$$\beta_i = 0.9, \quad \beta_u = 0.9 \text{ for } \frac{E}{U_m} = 0, \tag{4a}$$

$$\beta_i = 1.41, \quad \beta_u = 1 \text{ for } \frac{E}{U_m} \approx 1. \tag{4b}$$

In order to obtain the analogous values of β_u for the mode of sinusoidal current at the rectifier inputs, we make use of the curves of current and voltage (Fig. 1b). The variation in the voltage $\underline{u}$ at the rectifier inputs can, with this, be presented by the following relationship:

$$u = E + (U_m - E)\sin\vartheta,$$
$$0 \leqslant \vartheta \leqslant n\pi \quad (n = 1, 3, 5, \ldots). \tag{5}$$

Since $|u| = |u_d|$, then the expression for the rectified voltage will have the same form as in (5), with the sole difference that n = 1, 2, 3, . . .

According to (5), the mean value of the voltage at the rectifier output will equal

$$U_d = E + \frac{1}{\pi}\int_0^\pi (U_m - E)\sin\vartheta = E + \frac{2}{\pi}(U_m - E). \tag{6}$$

The effective value of the voltage at the rectifier input is

$$U = \sqrt{E^2 + \frac{(U_m - E)^2}{2}}. \tag{7}$$

By substituting (6) and (7) in (3) we obtain, after some transformations, the expression for the coefficient β_u for the mode of sinusoidal current at the rectifier inputs:

$$\beta_u = \frac{\frac{E}{U_m} + \frac{2}{\pi}\left(1 - \frac{E}{U_m}\right)}{\sqrt{\left(\frac{E}{U_m}\right)^2 + \frac{\left(1 - \frac{E}{U_m}\right)^2}{2}}}. \tag{8}$$

For a sinusoidal current at the input of a single-phase rectifier bridge, the coefficients β_u and β_i have the following values:

$$\beta_u = 0{,}9 \text{ for } \frac{E}{U_m} = 0, \tag{9a}$$

$$\beta_u = 1 \text{ for } \frac{E}{U_m} = 1, \tag{9b}$$

$$\beta_i = 9 \left(0 \leqslant \frac{E}{U_m} \leqslant 1\right). \tag{9c}$$

Due to the inconstancy of the coefficients β_u and β_i for a variable ratio of E/U_m, and also for sinusoidal current and sinusoidal voltage at the rectifier inputs, it is much more convenient, in designing the circuit, to use the replacement circuit with an equivalent active load r_e instead of the rectifier bridge.

For operation on a counter emf we have

$$U_d = E + I_d r, \tag{10}$$

where $\underline{r}$ is the active impedance in series with the counter-emf E (if the rectifier output is connected to the armature circuit of a motor or to a storage battery, then $\underline{r}$ is the active impedance of the armature winding or the battery's internal impedance).

The rectified voltage U_d can also be expressed in terms of the resistance r_d':

$$U_d = I_d r_d', \tag{11a}$$

where, in accordance with (10)

$$r_d' = \frac{E}{I_d} + r. \tag{11b}$$

The power applied to the rectifier bridge equals

$$P = UI. \tag{12}$$

By substituting (2) and (3) in (12), and taking (11a) into account, we get

$$P = I_d^2 \frac{r_d'}{\beta_u \beta_i}. \tag{13}$$

We introduce the concept of the equivalent load impedance r_e, and express the power P in terms of it

$$P = I^2 r_e, \tag{14}$$

where $r_e = U/I$.

By substituting (2) in (14), we get

$$P = I_d^2 \frac{r_e}{\beta_i^2}. \tag{15}$$

By equating (13) and (15), we obtain the definitive expression for the equivalent impedance r_e in the form

$$r_e = \frac{\beta_i}{\beta_u} r_d', \tag{16}$$

Figure 2 shows the circuit for a single-phase rectifier bridge operating on a counter-emf (a) and its equivalent circuit with active impedance r_e(b). Such a transfiguration of rectifiers with counter-emf to an equivalent active impedance significantly simplifies the circuit design if a

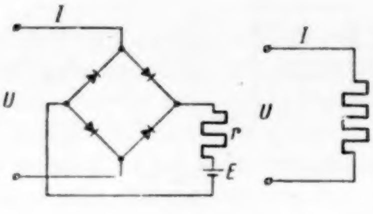


Fig. 2.

magnetic amplifier is connected in series with this impedance. For example, if one ignores the losses in the steel of the magnetic amplifier's cores, then the following equation will be valid for the fundamental harmonics of the voltages and currents (in order to simplify the notation, the subscripts for the fundamental harmonics have been suppressed in this and the following equations):

$$U_L^2 + I^2(r_e + r_{ma})^2 = U^2, \tag{17}$$

where U_L is the fundamental harmonic of the effective value of the inductive component of the voltage across the terminals of the magnetic amplifier's working winding $w_\sim$, U is the fundamental harmonic of the effective value of the source voltage, r_{ma} is the resistance of working winding $w_\sim$ of the magnetic amplifier.

By expressing the voltage and current in terms of the magnetic induction and magnetic field strength we obtain after some uncomplicated transformations,

$$\frac{B_m^2}{B_{m0}^2} + \frac{H^2}{H_0^2} = 1, \tag{18}$$

where

$$B_m = \frac{U_L}{4.44 f w_\sim S}, \quad B_{m0} = \frac{U}{4.44 f w_\sim S};$$

$$H = \frac{w_\sim I}{l}, \quad H_0 = \frac{w_\sim I_{max}}{l} = \frac{w_\sim U}{l(r_e + r_{ma})}. \tag{19}$$

Here, l and S are the average length and cross-section of the magnetic amplifier's magnetic circuit and I_{max} is the maximum current, corresponding to $U_L = 0$.

Equation (18) is the equation of an ellipse with its center at the origin and with semiaxes B_{m0} and H_0. By joining such an ellipse with the characteristics $B_m = f(H, H_=)$, we easily obtain the magnetic amplifier characteristics we are seeking. A graphicoanalytic method of obtaining magnetic amplifier characteristics is explained in more detail in [5].

The magnitude of the resistance r_e which enters into the expression for the magnetic field strength H_0 depends on the ratio of the coefficients B_i/B_u. For example, for a sinusoidal voltage, according to (4a) and (4b), we have: for $E/U_m = 0$, $r_e = r'_d$; for $E/U_m \approx 1$, $r_e = 1.41r'_d$. For sinusoidal current, according to (9a), (9b), and (9c), we have: for $E/U_m = 0$, $r_e = r'_d$; for $E/U_m = 1$, $r_e = 0.9r'_d$.

Thus, in determining the characteristics of a magnetic amplifier by conjoining the magnetic characteristics $B_m = f(H, H_=)$ with the equation of an ellipse, (18), three limiting cases can be found (Fig. 3): $E/U_m = 0$, i.e.,

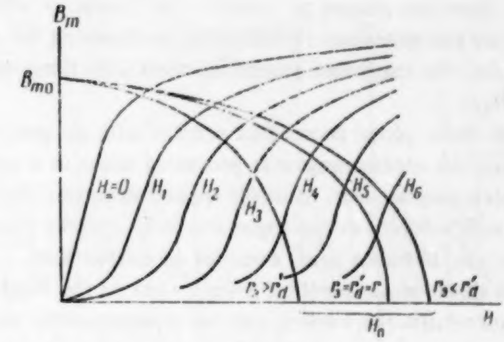


Fig. 3. Working trajectories of a magnetic amplifier with a counter-emf.

there is a purely active load ($r_e = r'_d = r_l$, where r_l is the load resistance); $E/U_m \approx 1$, i.e., the load is a counter-emf. Here there can be either $r_e = 0.9r'_d$ or $r_e = 1.41r'_d$.

In charging storage batteries, principally starter batteries, there is a significantly large voltage drop from their emf, caused by internal impedances, so that $E/U_m \approx 1$. In schemes for controlling the rotational speed of motor armatures, there can be, as a function of its magnitude, $0 \leq E/U_m \leq 1$.

If the voltage drop across a magnetic amplifier is greater than the voltage drop across the input (the magnetic amplifier is operating on the upper portion of the magnetic characteristics) (Fig. 3), then $r_e < r'_d$.

The method just presented for obtaining the desired characteristic is convenient in those cases when the supply voltage and the load parameters are invariant.

When the load parameters vary, it is convenient to give different values to the resistance r_e and the current I, and to compute the value of H corresponding to them from (19), and the value of B_m from the formula

$$B_m = \frac{\sqrt{U^2 - I^2(r_e + r_{ma})^2}}{4.44 f w_\sim S}. \tag{20}$$

By conjoining the thus obtained function $B_m = f(H)$ with the characteristics $B_m = f(H, H_=)$, we can find the magnetic amplifier characteristics being sought.

In many cases, the magnetic amplifier operates with H = const. Such modes of amplifier operation occur, for example, in charging starter storage batteries, in controlling the rotational velocity of a motor armature with constant torque, etc.

In order to obtain the characteristic of a saturated reactor magnetic amplifier, providing a constancy of charging current, it is necessary to know the law of variation of the voltage across the terminals of the storage battery as a function of the charging time for I = const, i.e., it is necessary to have the function $U_d = f(t)$ or $E = f(t)$ and the internal resistance r.

Magnetic Amplifier Operation with Three-Phase Rectifier Bridge

Three-phase rectifier schemes are widely used in technology, chiefly in power rectification stands. How-

ever, there are almost no works in the literature which consider the questions of designing, or choosing the parameters for, the magnetic amplifiers used with three-phase rectifiers.

In three-phase controlled circuits with magnetic amplifiers, the electromagnetic processes occur in a more complex manner than in single-phase circuits. The currents and voltages at the input and output of the rectifier bridge are different here, even for an active load.

A three-phase rectifier circuit, just as the single-phase circuits considered earlier, can be represented by an equivalent circuit. For calculating the circuit elements, it is convenient to represent a three-phase rectifier circuit by a Y-network. With this, the transfiguration must be so effected that, in the equivalent Y-network, the same linear voltages and currents occur as in the rectifier circuit, while the same power as before acts on the load.

With an active load, the power of one phase of the equivalent Y-network, P_p, equals

$$P_p = U_p I \tag{21a}$$

or

$$P_p = I^2 r_p, \tag{21b}$$

where r_p is the phase resistance of the equivalent Y-network.

In a circuit with an active load, the rectified voltage ordinarily equals

$$U_d = I_d r_d, \tag{22}$$

where r_d is the active load impedance.

By setting $U = U_p$ in (3), and by substituting (2), (3), and (22) in (21a) and (21b), we get

$$P_p = I_d^2 \frac{r}{\beta_u \beta_i} \tag{23a}$$

$$P_p = I_d^2 \frac{r_p}{\beta_i^2}. \tag{23b}$$

Equating (23a) and (23b), we obtain, finally,

$$r_p = \frac{\beta_i}{\beta_u} r_d. \tag{24}$$

Expression (24) is the condition for the transfiguration of a three-phase rectifier circuit to an equivalent Y-network with the resistance of each phase equal to r_p. This expression is analogous to Expression (16) for single-phase rectifier circuits.

Figure 4 shows the three-phase rectifier bridge circuit with an active load (a) and the equivalent Y-network corresponding to this circuit (b). An analogous Y-network is also equivalent to a three-phase circuit with zero output. The difference is solely in the magnitude of phase resistance r_p, determined by the coefficients β_i and β_u.

The values of the coefficients β_i and β_u for a three-phase rectifier bridge circuit operating on an active load are given in Table 1.

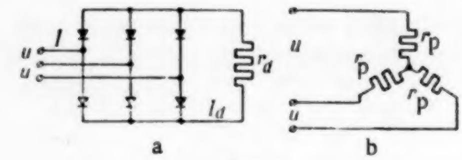


Fig. 4.

TABLE 1

Mode of rectifier operation	β_i	β_u
Sinusoidal voltage on rectifier input	1.227	2.12
Sinusoidal current on rectifier input	1.35	2.34

For other forms of load (counter-emf and active-inductive load), the coefficients β_i and β_u are given in [1, 6, 7].

Figure 5 shows the circuit of a three-phase rectifier with a magnetic amplifier and an active load (a) and the equivalent Y-network (b) (for the purposes of simplification, the functional dependence of the magnetic amplifier's induction on current is not shown in Fig. 5b).

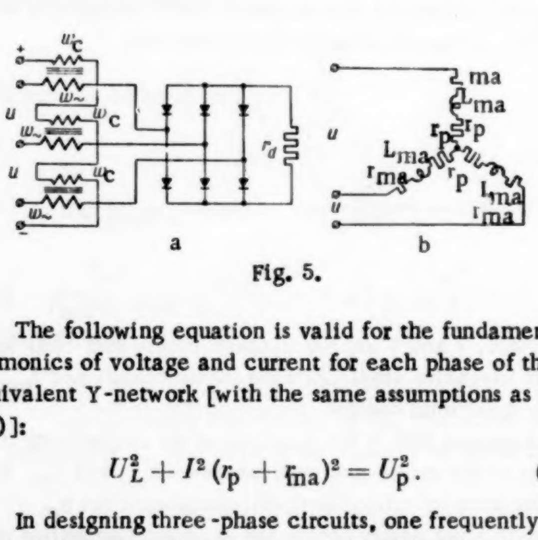


Fig. 5.

The following equation is valid for the fundamental harmonics of voltage and current for each phase of the equivalent Y-network [with the same assumptions as for (17)]:

$$U_L^2 + I^2 (r_p + r_{ma})^2 = U_p^2. \tag{25}$$

In designing three-phase circuits, one frequently has to do with linear voltage sources. By taking this into account, and by expressing the voltage and current in terms of magnetic induction and magnetic field strength, according to (19), after some uncomplicated transformations, we obtain

$$\frac{3B_m^2}{B_{m0}^2} + \frac{3H^2}{H_0^2} = 1. \tag{26}$$

For the substituting of (19) in (25), it is assumed that U is the linear source voltage and that $r_e = r_p$.

As with (18), (26) is the equation of an ellipse which, when joined to the characteristics $B_m = f(H, H_=)$, allows one to obtain easily the desired magnetic amplifier characteristics.

The coefficients β_i and β_u are different for the modes of sinusoidal current and sinusoidal voltage on the rectifier input. However, the ratios of the coefficients, β_i / β_u for each of these modes are almost identical, being equal to 0.579 for the mode of sinusoidal voltage, and to 0.577 for the mode of sinusoidal current.

Sinusoidal voltage at the rectifier input occurs only for $B_m = 0$. But, $B_m \neq 0$ always for magnetic amplifier operation. Therefore, in computing the magnetic amplifier characteristics $H = f(H_=)$, the ratio $r_p = 0.577 r_d$ should be used.

If it is required to determine the load characteristics $H_d = f(H_=)$, then it is necessary to multiply the ordinates of the curves $H = f(H_=)$ by the coefficient β_i.

For variations of the load parameters or of the line voltage U, it is convenient to compute the magnetic induction B_m by the formula

$$B_m = \frac{\sqrt{U^2 - 3I^2(r_p + r_{ma})^2}}{\sqrt{3}\cdot 4.44 f w_{\sim} S}. \tag{27}$$

For operation of the rectifier on a counter-emf, the equivalent Y-network in accordance with (11a), will have a form analogous to that shown in Fig. 4. The only difference will be that, in this case,

$$r_p = \frac{\beta_i}{\beta_u} r_d'. \tag{28}$$

For operation of a three-phase rectifier on a counter-emf, the same ratio is retained between the fundamental harmonic of the input voltage and the dc component of the output voltage as for a purely active impedance, i.e., for charging storage batteries with rectified current, the coefficients β_u given in Table 1 are valid. Also retained is the ratio between the fundamental harmonic of input current and the dc component of the output current [1]. As for an active load, we consider that $r_p = 0.577 r_d'$, and r_d' is defined by (22), where $r_d = r_d'$.

Schemes for Devices for the Automatic Stabilization of Charging Current

Charging of many types of storage batteries is carried out with dc which is invariant in magnitude. In many case, the invariance in current magnitude is necessary in the charging process in order that the battery charge can be gauged. For example, the charge on acid storage batteries is determined by the density of electrolyte and voltage across the terminals, both invariant over several hours, for one and the same charging current. The charge of alkaline batteries is determined by the magnitude of the capacitance obtained by them (with an invariant charging current, the charge is here determined actually by the batteries' charging time).

In charging devices with saturated-core magnetic amplifiers, it is not difficult to implement automatic stabilization of the charging current (shown in Figs. 6 and 7).

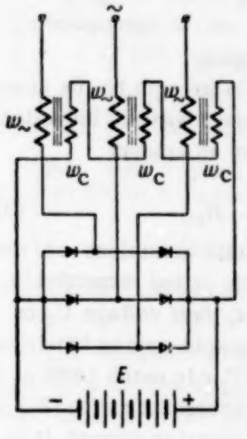


Fig. 7.

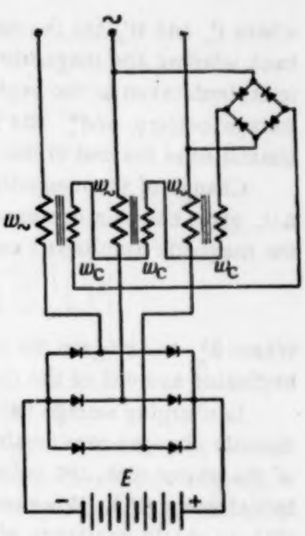


Fig. 6.

As a function of the quality of the steel and of the operational mode of the magnetic amplifier, the control winding w_c, in order to obtain a stabilized charging current, can be connected either to the terminals of the storage battery to be charged, to provide positive voltage feedback (Fig. 6), or to a source of invariable voltage, to provide constant bias (Fig. 7).

To explain the choice of one scheme or another for magnetic amplifier control, Fig. 8 shows the characteristics $B_m = f(H, H_=)$.

In the process of charging storage batteries, their voltages is changed from U_d' to U_d'' by the amount

$$\Delta U_d = U_d'' - U_d'. \tag{29}$$

This change of voltage, in the circuit with positive feedback (cf., Fig. 6), gives rise to an increase in the magnetic amplifier's magnetizing current and, consequently, the magnetic field strength due to the current in the positive feedback winding is increased by an amount $\Delta H_= \equiv \Delta U_d$. The quantity $\Delta H_=$ is defined by the formula

$$\Delta H_= = H_='' - H_=' = \frac{\Delta U_d}{E}\frac{w_c}{l} = (I_='' - I_=')\frac{w_c}{l}. \tag{30}$$

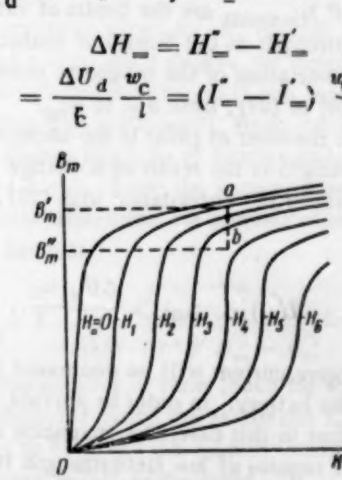


Fig. 8. Working trajectories of a magnetic amplifier used in charging devices.

where $I'_=$ and $H'_=$ are the magnetizing current of the feedback winding and magnetic field strength corresponding to it, both taken at the beginning of the charging of the storage battery, and $I''_=$ and $H''_=$ are the corresponding quantities at the end of the charging.

Change of the magnetic field strength by the amount $\Delta H_=$ gives rise to a decrease in the magnetic induction in the magnetic amplifier's cores by an amount

$$\Delta B_m = B'_m - B''_m, \qquad (31)$$

Where B'_m and B''_m are the magenetic inductions at the beginning and end of the charging period respectively.

In charging storage batteries, their voltage U_d ordinarily changes considerably (in acid storage batteries of the starter type, the voltage U''_d can reach 135% of its initial value U'_d). However, in accordance with (20) or (27), to obtain constancy of the charging current, it is necessary to decrease insignificantly the magnetic induction in the magnetic amplifier's cores. Therefore, the magnetic amplifier must operate in the saturated region of the characteristic $B_m = f(H, H_=)$ so that, with an increase in voltage U_d, the relative growth in the magnetic field strength $\Delta H_=$, induced by the positive feedback current, will be larger than the relative decrease in magnetic induction ΔB_m. In Fig. 8, the working trajectories of the magnetic amplifier are shown by the dashed lines between points a and b.

To obtain a charging current which is invariant in magnitude, it is necessary to choose such a mode of magnetic amplifier operation in which, for a variation in magnetizing magnetic field strength defined by (30), the magnetic induction must be changed by the amount ΔB_m, computed in accordance with (20) or (27) and (31). This mode of magnetic amplifier operation, providing automatic stabilization of the charging current, we express by the equation

$$(H''_= - H'_=)_{H=\mathrm{const}} = \frac{\Delta U_d}{r_c}\frac{w_c}{l}, \qquad (32a)$$

where $(H''_= - H'_=)_{H=\mathrm{const}}$ are the limits of variation of the magnetizing strength on the family of characteristics $B_m = f(H, H_=)$ for variation of the magnetic induction, computed from (20) or (27), from B'_m to B''_m.

The right member of (32a) is the increment of magnetic field strength as the result of a change in voltage of the storage battery in accordance with (29).

If

$$(H''_= - H'_=)_{H=\mathrm{const}} > \frac{\Delta U_d}{r_c}\frac{w_c}{l}, \qquad (32b)$$

then the charging current will be decreased in the process of charging the battery. In order to provide a constant charging current in this case, the magnetic amplifier must operate in the regions of low field strength H or high field strength $H_=$, i.e., the trajectory must be shifted either to the left, or down, or down and to the left simultaneously.

then the charging current will be increased in the process

$$(H''_= - H'_=)_{H=\mathrm{const}} < \frac{\Delta U_d}{r_c}\frac{w_c}{l}, \qquad (32c)$$

of charging the battery. In order to provide here a constant charging current, the magnetic amplifier must operate in the regions of large field strength H and low field strength $H_=$, i.e., the trajectory must be shifted either to the right, or up, or up and to the right simultaneously.

Charging devices in which stabilization of charging current was provided by the choice of the proper mode of magnetic amplifier operation were considered in [3, 4].

To decrease the dimensions and weight of the power magnetic amplifiers used in charging devices, it is ordinarily recommended that one use cold-rolled steel such as, for example, É320, É330, É370. These brands of steel differ from transformer steel, not only by their large values of magnetic saturation induction, but also by the steepness of the slope of the characteristics, particularly in the case when toroidal cores prepared from strips are used [8].

If the characteristics $B_m = f(H, H_=)$ of the magnetic amplifiers used are almost vertical in their unsaturated portions then, to provide constancy of charging current, it suffices to provide a constant bias using, for this, small-size germanium diodes of type DG-Ts (Fig. 7). Connecting a battery in the bias circuit can permit the elimination of the effect of oscillations in the supply voltage U on the magnitude of the charging current. Since the magnetic induction B_m changes within narrow limits during the battery charging process, then this method allows the implementation of a completely satisfactory stabilization of charging current.

Example of the Determination of the Basic Parameters of a Saturated Reactor Magnetic Amplifier

We earlier considered methods for determining the characteristics of magnetic amplifiers operating on a counter emf. We now show that, by means of the relationships already provided, we can also compute the basic parameters of a planned magnetic amplifier.

We consider the sequence of the determinations of the basic parameters of a magnetic amplifier intended for the charging of 14 starter storage batteries of type 6STÉN-140M by a current of invariable magnitude, equal to 10 amperes.

In Table 2 (the upper two rows) there are given the charging characteristics of one element of a 6STÉN-140M storage battery, 50% discharged, for charging by a 10 amp current.

The rectified voltage will equal

$$U_d = mnU_{el},$$

where m is the number of storage batteries to be charged (m = 14) and n is the number of elements in one battery (n = 6).

To determine whether or not it is possible to charge the given quantity of storage batteries, one must com-

TABLE 2

Circuit parameters	Time, in hours						
	0	1	2	4	6	8	10
Charging current I_d amperes	10	10	10	10	10	10	10
Voltage on a storage battery element, volts	2.1	2.17	2.2	2.23	2.34	2.59	2.59
Rectified voltage, volts	176.4	182	185	187	196.4	217.5	217.5
Equivalent phase resistance r_p, ohms	10.18	10.5	10.68	10.8	11.33	12.54	12.54
$B_m = f(t)$	. . .	. . .	. . .	. . .	. . .	. . .	. . .

pute $U_{d\,max}$. For this, it is necessary to know the maximum possible voltage on one element, $U_{el\,max}$. For starter batteries, ordinarily $U_{el\,max} \leq 2.7$ v. Then, $U_{d\,max} = 226.8$ v.

According to (3) and the data of Table 1, in the example under consideration, if the voltage drop in the rectifier is neglected, the rectified voltage can equal

$$U_d = \beta_u U_p = (2.12 - 2.34)\frac{220}{\sqrt{3}} = 269 - 297\text{v}.$$

With the given relationships of the possible rectified voltage and the maximum possible voltage on the storage batteries, we decide that charging will occur with an approximately sinusoidal voltage on the rectifier bridge's input. But since $B_m \neq 0$ with this, then the equivalent phase resistance will be defined by the relationship

$$r_p = 0.577\frac{U_d}{10}.$$

The results of computing r_p are given in Table 2.

The current I on the rectifier input, according to (2), will equal 8.15 amp.

In (27), which relates the given parameters with the magnetic amplifier's parameters, there are still four unknown quantities: B_m, $w_\sim$, S and r_{ma}. The active impedance of winding $w_\sim$ is very small and, for a preliminary estimate of the parameters, it may be neglected.

When positive voltage feedback is used, constancy of charging current can be attained only when the magnetic amplifier operates on the saturated portion of characteristics $B_m = f(H, H_=)$. Therefore, the magnitude of B_m can be given for a well-known brand of steel. By being given the magnitude of S as well, we can compute the number of turns of winding $w_\sim$.

In order that the initially chosen values of $w_\sim$ and S correspond closely to the actual magnitudes, we can use the following assumptions.

For the initial and final moments of storage battery charging, the following relationships are valid:

$$U^2 = U'^2_L + I^2 r'^2_p, \qquad U'_L = 4.44 f w_\sim S B'_m \cdot 10^{-8},$$
$$U^2 = U''^2_L + I^2 r''^2_p, \qquad U''_L = 4.44 f w_\sim S B''_m \cdot 10^{-8},$$
$$H' = H'' = H = \frac{I w_\sim}{l}.$$

The quantities with one prime (') relate to the beginning of the charging period of the storage battery, while those with two primes relate to the period's end. The resistances r_{ma} are ignored here, since we are making only a preliminary choice of parameters.

By transforming these equations, we obtain a function for the determination of the volume of steel

$$V = \frac{I^2\sqrt{r''^2_p - r'^2_p}}{4.44 f H \cdot 10^{-8}\sqrt{B'^2_m - B''^2_m}}.$$

Due to the increase of voltage on the storage batteries to provide I = const, the magnetic induction B_m must be decreased. This decrease in induction is ordinarily insignificant in the charging of starter storage batteries. By taking this into account, and by selecting the magnetic induction B_m, one can compute the value of the square root $\sqrt{B''^2_m - B'^2_m}$.

By being given the magnitude of the magnetic field strength H for the selected type of steel, we can approximately determine the core volume.

If the invariance of the charging current is to be provided by biasing, then the chosen magnitude of H must correspond to the vertical segments of the steel's magnetic characteristics.

The expression given for the volume of steel can facilitate to a significant degree the determination of the magnetic amplifier parameters if one uses, for example, standard three-leg laminas [9]. From the known volume V one can choose the type of lamina and, from this, one can select l and S. From the known value of S, one can approximately determine the number of turns of winding $w_\sim$ by the formula

$$w_\sim = \frac{I\sqrt{r''^2_p - r'^2_p}}{4.44 f S \cdot 10^{-8}\sqrt{B'^2_m - B''^2_m}}.$$

From the number of turns found for winding $w_\sim$, one can approximately gauge the resistance r_{ma}, for which it is necessary to be given the current density and to compute the mean length of the turns.

In thus computing approximately the basic parameters of the magnetic amplifier, it is necessary to deter-

mine the law of variation of variation of magnetic induction $B_m = f(t)$ in accordance with (27) and to conjoin the function thus obtained for H = const with the characteristics $B_m = f(H, H_=)$. The results of the computation of $B_m = f(t)$ can be found in Table 2.

The choice of the required mode of magnetic amplifier operation to provide I_d = const was presented in a previous section.

Limitations on the size of this paper prevent us from giving here the entire design of the magnetic amplifier.

CONCLUSIONS

The transfiguration suggested of one-phase and three-phase rectifier bridges to equivalent loads permits the use of the simplest, most graphic and sufficiently accurate graphicoanalytic method of designing saturated reactor magnetic amplifiers.

In this paper we have shown the simplest methods of stabilizing charging current, which are attained by the proper choice of operating mode of the magnetic amplifier.

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ON THE SIMULATION OF LINEAR SYSTEMS WITH n STATIONARY RANDOM INPUT PROCESSES WITHOUT THE USE OF RANDOM FUNCTION GENERATORS

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A method is described for determining the mean value of the square of the output quantity of a linear system with n stationary random input processes by means of a simulating device without the use of random function generators. This method is a generalization to systems with n inputs of the method described in [2].

In solving various problems in the domain of regulation and automatic control, one frequently has to deal with the problem of determining the average value of the square of the output quantity of a system with n stationary random processes at the system's inputs. We shall assume that the input processes are stationary and ergodic, and that their spectral densities (proper and mutual) are rational fractional functions of the complex frequency s.

Posing of the Problem

We denote by y(t) the output quantity of the linear system under consideration. The mean value of the square of y(t) is defined by the relationship

$$\overline{y^2} = \lim_{t\to\infty} \frac{1}{t} \int_0^t [y(\tau)]^2 d\tau. \qquad (1)$$

A generator of n stationary random processes with given proper and mutual spectral densities was described in [1]. On the basis of this generator we can construct a block schematic for determining the mean value of the square of the system's output quantity (Fig. 1). On this block schematic, $Q_1, Q_2, \ldots, Q_n$ are the noncorrelated generators of white noise, Sh are shaping filters, S_n is the given linear system with n inputs, Sq is a squaring device, I is an integrator, $x_1(t), x_2(t), \ldots, x_n(t)$ are the random input processes of system S_n, and y(t) is the system's output process.

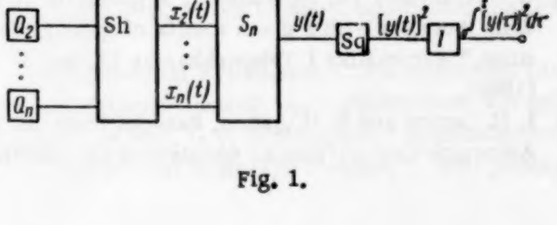

Fig. 1.

For sufficiently large t, one can determine the mean value of the square of y(t) from the integrator's output quantity in accordance with (1). With this, it is necessary to use n generators of noncorrelated white noise. These generators must be quite accurate and stable.

A method was described in [2], applicable to linear systems with one stationary random input process, for determining the mean value of the square of the output quantity by means of a simulating device, eliminating the necessity of using white noise generators. This method is based on the possibility of generating an input random process with a known spectral density by means of a white noise generator and a linear shaping filter Sh (the determination of whose transfer function is described in [1, 2,]).

We denote by W(t) the impulsive response of the series-connected shaping filter Sh and the given linear system S_1. Based on known relationships, we obtain

$$\overline{y^2} = \int_0^\infty [W(\tau)]^2 d\tau. \qquad (2)$$

The block schematic shown on Fig. 2 follows from this. In this case, the input of the shaping filter has applied to it, not white noise, but a unit pulse function (a Dirac function). With increasing t, the integrator's output quantity tends to the quantity $\overline{y^2}$.

The superiority of this method is primarily that the white noise generator is excluded from the solving block schematic. The linear system is acted upon by a precisely defined function of time, which increases the accuracy of the determination of the quantity $\overline{y^2}$. Below we give a generalization of this method for a linear system S_n with n stationary random process at the system's inputs.

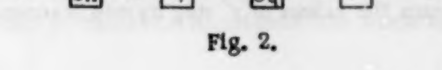

Fig. 2.

Solution of the Problem

The correlation function of the output quantity of a linear system with n stationary random processes at the inputs is defined [2] by the relationship.

$$\varphi_{yy}(\tau) = \sum_{m=1}^{n} \sum_{k=1}^{n} \int_0^\infty \left[W_m(\tau_1) \int_0^\infty W_k(\tau_2)\, \varphi_{mk}(\tau - \tau_2 + \tau_1)\, d\tau_2\right] d\tau_1, \tag{3}$$

where $\varphi_{mk}(\tau)$ is the mutual correlation (cross-correlation) function of the mth and kth random processes at the input, $W_i(\tau)$ is the system's impulsive response to the ith input and $\varphi_{yy}(\tau)$ is the correlation function of the system's output quantity.

We use (3) to determine the magnitude of $\overline{y^2}$. Since the series connection of the shaping filters Sh and the system S_n is a linear system with n inputs then, in our case, $W_i(\tau)$ will be the impulsive response to the ith input of shaping filters Sh at the output of the system S_n. The function $\varphi_{mk}(\tau)$ (the correlation functions of the random processes at the inputs of shaping filters Sh) are defined by the equations

$$\begin{aligned} \varphi_{mk}(\tau) &= \delta(\tau) \quad \text{for} \quad m = k, \\ \varphi_{mk}(\tau) &= 0 \quad \text{for} \quad m \neq k. \end{aligned} \tag{4}$$

By substituting (4) in (3), for $\tau = 0$, we obtain the quantity $\overline{y^2}$ in the form

$$\overline{y^2} = \varphi_{yy}(0) = \sum_{i=1}^{n} \int_0^\infty [W_i(\tau)]^2\, d\tau. \tag{5}$$

We denote by $\overline{y_i^2}$

$$\overline{y_i^2} = \int_0^\infty [W_i(\tau)]^2\, d\tau, \tag{6}$$

i.e., the mean value of the square of the output quantity y when the inputs of the shaping filters are acted upon by only one white noise generator Q_i.

Then, from (5), we get

$$\overline{y^2} = \sum_{i=1}^{n} \overline{y_i^2}. \tag{7}$$

Consequently, each value of $\overline{y_i^2}$ [cf., (6)] can be determined by the method described in [2].

Thus, the mean value of the square $\overline{y^2}$ of the output quantity of linear system S_n with n stationary random input processes can be obtained at the output of the block schematic shown on Fig. 3. At the inputs of shaping filters Sh there act, at the individual moments of time T_i, unit pulse functions $\delta(t-T_i)$. The moments $T_i (i = 1, 2, \ldots, n)$ must be so determined that, at these moments, the entire system is found in a virtual resting state. The integrator I sums the values of $\overline{y^2}$ step by step in accordance

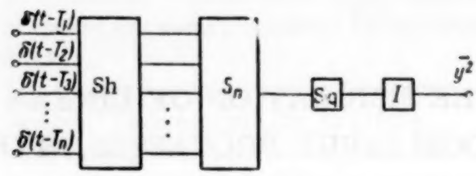

Fig. 3.

with (7), such that, for $t > \max\{T_1, T_2, \ldots, T_n\}$, its output quantity tends to the value $\overline{y^2}$.

It is well known, from the methods of solving problems on simulating devices (analog computers), that the unit pulse functions can be replaced by the introduction of the proper initial conditions.

CONCLUSIONS

In the solution of various practical problems, it is necessary to determine the mean value of the square of the output quality of a linear system S_n at the n inputs of which there act n stationary random processes with known spectral densities in the form of rational-fractional functions of the complex frequency s. This problem can be solved by using the block schematic shown in Fig. 1.

For a system S_1 with one stationary random process at the input, a method is described in [2] which eliminates the use of a random process generator. In the present paper, this method was generalized to an arbitrary linear system S_n with n inputs, acted upon by n stationary random processes with known proper and mutual spectral densities.

The block schematic for the analog computer computations consists, in addition to the given linear system, of shaping filters (their transfer functions are determined by the very simple methods described in [1]) a squaring device and an integrator (Fig. 3). At the inputs of the shaping filters appear, not white noise, but unit pulse functions. With this, the mean value of the square $\overline{y^2}$ is obtained at the integrator's output.

The advantage of this method of computing the quantity $\overline{y^2}$ is, above all, that random function generators are eliminated. Since the entire system is acted upon, not by random processes, but by accurately defined functions of time [the pulse functions $\delta(\tau)$], one can expect a greater accuracy in determining the desired quantity $\overline{y^2}$.

LITERATURE CITED

[1] I. Matyash and Ya. Shilkhanek, "A generator of random processes with a given matrix of spectral densities," Avtomatika i Telemekhanika 21, No. 1 (1960).

[2] J. H. Laning and R. H. Batten, Random Processes in Automatic Control [Russian translation] (IL, 1958).

INFORMATION

BUREAU OF THE PRESIDIUM OF THE ACADEMY OF SCIENCES OF THE USSR

Decision of Feb. 12, 1960, Number 134, Moscow

On the Decision of the Conference on the Theory of Invariance and its Application to Automatic Devices of October 20, 1958 (Kiev)

After hearing the communication of the chairman A. A. Dorodnitsyn, of the commission of the Presidium of the AN SSSR on the decision of the Conference on the Theory of Invariance and its Application to Automatic Devices, of Oct. 20, 1958 (in Kiev), the Bureau of the Presidium of the Academy of Science of the USSR has decided:

1. To endorse the conclusions of the commission of the Presidium of the AN SSSR on the decision of the Conference on the Theory of Invariance and its Application to Automatic Devices, of Oct. 20, 1958 (in Kiev) the Appendix).
2. To publish the conclusions of the commission in the journal Avtomatika i Telemekhanika (Automation and Remote Control).

Vice-President of the
Academy of Science, USSR
Academician A. V. Topchiev

Chief Scientific Secretary
Presidium of the Academy of
Science, USSR
Corresponding Member AN SSSR
E. K. Fedorov

APPENDIX TO THE DECISION OF THE BUREAU OF THE PRESIDIUM AN SSSR OF FEBRUARY 12, 1960, Number 134

Conclusions of the Commission in Connection with the Discussion on the Theory of Invariance

The commission, consisting of Academician A. A. Dorodnitsyn, Academician AN Ukrainian SSR A. Yu. Ishlinskii and Corresponding Member AN SSSR B. N. Petrov, designated on October 28, 1958 by the vice president of the AN SSSR, Academician A. V. Topchiev, in considering the decisions of the Kiev Conference on the Theory of Invariance and Its Application to Automatic Devices (Oct. 20, 1958), the decisions of the Presidium of the Academy of Science of the USSR of April 1, 1941, Section 5, the conclusions of the commission of the Presidium of the Academy of Science of the USSR for discussion and judging the work of Professor G. V. Shchipanov, "Autocontrolled systems with several degrees of freedom," the work of Prof. G. V. Shchipanov itself, and certain materials in the discussion of the work of G. V. Shchipanov, established the following.

In the journal Avtomatika i Telemekhanika (Automation and Remote Control) No. 1 (1939) there was published the paper of Prof. G.V. Shchipanov, "Autocontrolled systems with several degrees of freedom". This paper gave rise to wide discussion, the materials of which were published in a number of journals.

By decision of the Presidium of the Academy of Science of the USSR, taken on March 4, 1940, a commission was created to discuss and judge the work of Prof. G. V. Shchipanov, "Autocontrolled systems with several degrees of freedom," this commission consisting of Vice-President of the AN SSSR Academician O. Yu. Shmidt, Academician S. A. Chaplygin, S. L. Sobolev, N. E. Kochin, and Corresponding Member AN SSSR N. G. Bruevich.

The commission's conclusions were discussed at the April 1, 1941 session of the Presidium of the AN SSSR, at which time the decision was taken to take the commision's conclusions under consideration, to publish them, and to consider the work of the commision as finished.

The commission's conclusions contained the dissenting opinions of Academicians V. S. Kulebakin and N. N. Luzin which considered necessary further research into the conclusions of Section 2, in which the estimate of the work of G. V. Shchipanov was contained.

In the following years, a number of works were published which, in some degree or other, treated the questions of compensation or invariance (the works of Academician

N. N. Luzin, Academician V. S. Kulebakin, A. G. Ivakhnenko, B. N. Petrov, G. M. Ulanov, and others).

On October 16-20, 1958, in Kiev, there was held the Conference on the Theory of Invariance and Its Application to Automatic Devices, convened by the Engineering Section of the AN Ukr SSR, the Kiev City Seminar and the Electrical Engineering Institute of the Ukr SSR, in which scientific workers from Moscow, Leningrad, Kiev and other cities of the Soviet Union participated.

At the Kiev Conference, a decision (resolution) was taken, noting the necessity of developing methods of disturbance compensation and the principle of invariance, in which it was stated, in Section 7:

"7. In view of the fact that the decision of the Presidium of the Academy of Science of the USSR on April 1, 1941 concerning the assertions of the conclusions of the special commission on the work of G. V. Shchipanov delayed development of the theory and practice of constructing automatic systems using compensation of disturbing stimuli and various invariance principles and does not make possible the proper and necessary progress in this important area of automation today, we ask the Presidium of the Academy of Science of the USSR to reconsider its decision as concerns the erroneous conclusions of the aforementioned commission."

As a result of the consideration of all the material cited above, we can draw the following conclusions:

1. The conclusions of the aforementioned commission to discuss and judge the work of Prof. G. V. Shchipanov, taken into consideration and published by the Presidium of the Academy of Science of the USSR, must be considered correct as concerns the first point and the first paragraph of the second point, in which it is stated that:

"1. The commission considers that questions of control theory have undoubtedly tremendous importance in the matter of the development of our national economy and the strengthening of the defense of our country, and must be worked out in the Institute for Automation and Remote Control of the Academy of Science of the USSR.

2. The work of Prof. G. V. Shchipanov under consideration, "Autocontrolled systems with several degrees of freedom," contains a whole series of incorrect and unfounded assertions."

Indeed, G. V. Shchipanov's work did contain incorrect and unfounded assertions. Among these assertions were the author's conclusions that, if the compensation condition held, then an ideal and universal controller was obtained; that the construction of regulators which did not satisfy the theorems presented in the paper must be considered as unsuitable; that with the compensation condition met, and for zero initial conditions, a free component appears, etc.

As is clear from the test and diagrams of G. V. Shchipanov's paper, he considered controllers based on the principle of deviation without additional direct or indirect stimuli proportional to the disturbance f(t) acting on the object of control, and he attempted to use, for the synthesis of such controllers, the condition of complete compensation while, in this case, it is impossible to use the condition of complete compensation.

2. However, the commission's conclusion (second paragraph of the second point) that the basic error in Prof. G. V. Shchipanov's paper is the "compensation condition", which leads mathematically to absurd conclusions and which does not correspond at all to actually realizable controllers, contains itself an erroneous conclusion, in that it is far too general a statement.

The basic error of G. V. Shchipanov consisted, not in the formulation of this condition, as was asserted in the commission's conclusions (on the contrary, this was his fundamental result!), but in the application of the compensation condition to the design of a class of control systems which does not permit its use.

As was shown by subsequent investigations, including those in Proceedings of the Kiev Conference on Invariance Theory, "the compensation condition" or the "invariance condition" can be used for designing a number of types of automatic control systems, for example, combined systems, with stimulus by deviation and by disturbance, interconnected automatic control systems for several quantities, and also several types of dynamic systems which do not come under the heading of automatic control systems, for example, bridge systems, complex electric networks, gyroscopic devices, etc.

Thus, the "compensation condition", or the "invariance condition" formulated by Prof. G. V. Shchipanov was a new design relationship which can be successfully used for designing a definite class of dynamic systems.

3. The aforementioned inaccuracy in the formulation of the commission's conclusions, and the absence of constructive criticisms in them, led, in some cases, to incorrect judgments on works in automatic control, even when the compensation condition was used correctly in them, which could not render possible development in this direction. Therefore, the commission considers it advantageous to publish, in one of the technical journals an expanded paper devoted to the prospective applications of the invariance principle, and which would cite the cases when it is advantageous to use this principle and when its use is inadmissible.

At the same time, the commission considers it necessary to note the imperfect editing of the decision of the Kiev Conference on the Theory of Invariance and its Application to Automatic Devices, from which one might gather the impression that the works of G. V. Shchipanov did not contain errors.

Commission Members
A. A. Dorodnytsin,
A. Yu. Ishlinskii,
B. N. Petrov.